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DIRICHLET SERIES[1]

I

GENERAL PROPERTIES OF CONVERGENCE

Let $\{\lambda_n\}(n \geqq 1)$ be a sequence of numbers all positive, except perhaps λ_1, which can be positive or zero, strictly increasing to infinity, and let $\{a_n\}$, $(n = 1, 2 \cdots)$, be a sequence of complex numbers.

The series

$$\sum_{n=1}^{\infty} a_n e^{-\lambda_n s}, \tag{1}$$

where $s = \sigma + it$ is a complex number (with σ and t real), is called a Dirichlet series.

If in a Taylor series

$$\sum_{n=1}^{\infty} a_n z^n \tag{2}$$

we put $z = e^{-s}$, the series becomes a Dirichlet series

$$\sum_{n=1}^{\infty} a_n e^{-ns} \tag{3}$$

which we shall call a *Taylor-D series*. Here $\lambda_n = n\,(n \geqq 1)$. Another example of Dirichlet series is furnished by the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} e^{-(\log n)s}, \tag{4}$$

[1] A series of lectures delivered at the Rice Institute during the academic year 1942–43 by S. Mandelbrojt, Docteur ès Sciences (Paris), Professor at the Collège de France, Visiting Professor of Mathematics at the Rice Institute.

In the present course the author does not pretend to give a general theory of Dirichlet series. This was masterfully done some ten years ago by Vladimir Bernstein (see bibliography [3]). Apart from certain elementary results concerning convergence of Dirichlet series, we give here general results, some very recent, based essentially upon methods introduced by the author in his previous papers.

which represents the famous $\zeta(s)$ function of Riemann. In this case $\lambda_n = \log n$. The function $\zeta(s)$, as well as the functions represented by the series of the more general form

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \sum_{n=1}^{\infty} a_n e^{-(\log n)s}$$

plays a very important rôle in the theory of numbers.

Let us recall that the Taylor series (2) admits a radius of convergence, R, which may be equal to 0, to infinity, or to a positive finite number. In the first case, the series converges only for $z=0$, in the second case it converges for every value of z, in the case when $0<R<\infty$ the series converges for $|z|<R$ and converges for no value z such that $|z|>R$. The circle $|z|<R$ (if $0<R$) is the circle of convergence of (2). In each case R is given by the formula:

$$\frac{1}{R} = \limsup_{n=\infty} |a_n|^{\frac{1}{n}}.$$

It is, moreover, well known that the series (2) converges absolutely inside the circle of convergence, and converges uniformly in each closed bounded region situated inside the circle of convergence.

Since the transformation $z=e^{-s}$ gives $\sigma = -\log|z|$, $t = -\operatorname{Arg} z$, we see that, if z varies on a curve L inside a circle $|z|<a$ the variable s varies on a curve L' which is situated in the half-plane $\sigma > \log a$. (If to a point z_0 of L we make correspond the point $s_0 = \sigma_0 + it_0$ with $0 \leqq t_0 < 2\pi$ the curve L' is well defined, since we agree to consider s as varying in a continuous way as z varies on L).

The series (3) converges therefore absolutely for

$$\sigma > \limsup_{n=\infty} \frac{\log|a_n|}{n} = \sigma_C, \tag{5}$$

if $\sigma_C < \infty$, and does not converge if $\sigma < \sigma_C$ (if $-\infty < \sigma_C$). The series (3) converges also uniformly in every closed bounded

region situated in the half-plane (5). But the series (3) does not converge at any point if $\sigma_C = \infty$.

The facts just mentioned cannot be translated directly into general Dirichlet series, in replacing simply the quantities n by λ_n. The most striking difference between general Dirichlet series and Taylor-*D* series, from the point of view of convergence, consists in the non-coincidence of the half-planes of different kinds of convergence.

The following theorem furnishes the greater part of results concerning convergence properties of a Dirichlet series (see for instance [**3**]):[1]

THEOREM I. *If* $\sum a_n e^{-\lambda_n s_0}$, $(s_0 = \sigma_0 + it_0)$ *converges, then* $\sum a_n e^{-\lambda_n s}$ *converges uniformly in every closed angle given by*

$$|\operatorname{Arg}(s - s_0)| \leqq \gamma < \frac{\pi}{2}.$$

Let us set $s = s_0 + s'$, $s' = \sigma' + it'$, $\sigma' > 0$, $|\operatorname{Arg} s'| < \gamma$, and let us write

$$\sum_1^m a_n e^{-\lambda_n s_0} = A_m(s_0) = A_m$$

$$\sum_1^\infty a_n e^{-\lambda_n s_0} = S = \lim_{m=\infty} A_m.$$

We have: $\sigma' > 0$, $\dfrac{|t'|}{\sigma'} \leqq \tan\gamma = M < \infty$.

We have for $s = s_0 + s'$ and $q > p \geqq 2$:

$$\sum_p^q a_n e^{-\lambda_n s} = \sum_p^q a_n e^{-\lambda_n s_0} e^{-\lambda_n s'}$$

$$(6) \quad = \sum_p^q (A_n - A_{n-1}) e^{-\lambda_n s'} = \sum_p^q [(A_n - S) - (A_{n-1} - S)] e^{-\lambda_n s'}$$

$$= \sum_p^{q-1} (A_n - S)(e^{-\lambda_n s'} - e^{-\lambda_{n+1} s'}) + (A_q - S) e^{-\lambda_q s'}$$

$$- (A_{p-1} - S) e^{-\lambda_p s'}.$$

[1]Numbers in brackets refer to the bibliography at the end of the Pamphlet.

If for $\epsilon>0$ given, p is chosen such as to have $|A_n-S|<\epsilon$ for $n\geqq p-1$, then

$$|\sum_p^q a_n e^{-\lambda_n s}| \leqq \sum_p^{q-1} |A_n-S|.|e^{-\lambda_n s'}-e^{-\lambda_{n+1}s'}|$$

$$+|A_q-S|e^{-\lambda_q\sigma'}+|A_{p-1}-S|e^{-\lambda_p\sigma'} \tag{7}$$

$$\leqq \epsilon\sum_p^{q-1} |e^{-\lambda_n s'}-e^{-\lambda_{n+1}s'}|+2\epsilon.$$

But

$$|e^{-\lambda_n s'}-e^{-\lambda_{n+1}s'}| = |s'|.\left|\int_{\lambda_n}^{\lambda_{n+1}} e^{-us'}du\right| \leqq |s'|\int_{\lambda_n}^{\lambda_{n+1}} e^{-u\sigma'}du$$

$$=\frac{|s'|}{\sigma'}(e^{-\lambda_n\sigma'}-e^{-\lambda_{n+1}\sigma'}) \leqq (M+1)(e^{-\lambda_n\sigma'}-e^{-\lambda_{n+1}\sigma'}),$$

and it follows from (7) that

$$|\sum_p^q a_n e^{-\lambda_n s}| \leqq 2\epsilon+\epsilon(M+1)\sum_p^{q-1}(e^{-\lambda_n\sigma'}-e^{-\lambda_{n+1}\sigma'})$$

$$=2\epsilon+\epsilon(M+1)(e^{-\lambda_p\sigma'}-e^{-\lambda_q\sigma'}) \leqq \epsilon(M+3).$$

This proves the theorem.

The following theorem is an immediate corollary of Theorem I.

THEOREM II. *If* $\sum a_n e^{-\lambda_n s_0}$ $(s_0=\sigma_0+it_0)$ *converges, the series* $\sum a_n e^{-\lambda_n s}$ *converges at each point* $s=\sigma+it$ *with* $\sigma>\sigma_0$, *and converges uniformly in each closed bounded region which is situated in the half-plane* $\sigma>\sigma_0$.

If we now replace the series (1) by the series

$$\sum_1^\infty |a_n| e^{-\lambda_n s}, \tag{8}$$

we see from Theorem II that if (8) converges for $s_0=\sigma_0$ (real), it converges also for $s=\sigma>\sigma_0$. Since $|e^{-\lambda_n s}|=e^{-\lambda_n\sigma}$, we see that the following theorem holds.

THEOREM III. *If the series* (1) *converges absolutely for* $s=s_0=\sigma_0+it_0$, *it converges absolutely for* $s=\sigma+it$ *with* $\sigma>\sigma_0$.

By a classical reasoning, in which the notion of Dedekind's

cuts is involved, Theorems II and III lead immediately to

THEOREM IV. *If there exists a point at which the series* (1) *converges, then either* (1) *converges for each value of* s *or there exists a real number* $\sigma_C(-\infty < \sigma_C < \infty)$ *such that* (1) *converges for* $\sigma > \sigma_C$ *and converges for no* s *with* $\sigma < \sigma_C$. *If* $\sum |a_n| e^{-\lambda_n \sigma}$ *converges for one value of* σ, *then either this series converges for all the values of* σ, *or there exists a quantity* $\sigma_A(-\infty < \sigma_A < \infty)$ *such that this series converges for* $\sigma > \sigma_A$ *and diverges for* $\sigma < \sigma_A$.

If the series (1) converges at no point we shall write $\sigma_C = \infty$. If (1) converges at every point we shall write $\sigma_C = -\infty$. If $\sum |a_n| e^{-\lambda_n \sigma}$ converges for no value of σ we shall write $\sigma_A = \infty$, and if it converges for each value of σ we shall write $\sigma_A = -\infty$.

The quantities σ_C and σ_A are called, respectively, *abscissa of convergence*, and *abscissa of absolute convergence* of the series (1). The straight-lines given by $\sigma = \sigma_C$, $\sigma = \sigma_A$ are called *axis of convergence* and *axis of absolute convergence* of the series. Obviously $\sigma_A \geqq \sigma_C$. For a Taylor-D series $\sigma_A = \sigma_C$, but this is not true in general. It is, for instance, known that the series $\sum \frac{(-1)^n}{n^s}$ converges for $\sigma > 0$ and does not converge for $\sigma \leqq 0$. On the other hand, $\sum \frac{1}{n^\sigma}$ converges for $\sigma > 1$ and diverges for $\sigma \leqq 1$. For the series $\sum \frac{(-1)^n}{n^s}$: $\sigma_C = 0$, $\sigma_A = 1$. The values of σ_C and σ_A are furnished by the following theorem.

THEOREM V. *Let us set*

$$a = \limsup_{n=\infty} \frac{\log |\sum_1^n a_m|}{\lambda_n}. \tag{9}$$

The series (1) *converges for* $\sigma > \max(0, a)$, *and if* $a > 0$, *then* $\sigma_C = a$. *Let us set*

$$b = \limsup_{n=\infty} \frac{\log \sum_1^n |a_m|}{\lambda_n}. \tag{10}$$

The series (1) *converges absolutely for* $\sigma > \max(0, b)$, *and if* $b > 0$, *then* $\sigma_A = b$.

The part concerning the absolute convergence follows immediately from the first part if we replace the series (1) by $\sum |a_n| e^{-\lambda_n s}$.

Let us set $A_n = A_n(0) = \sum_1^n a_m$. Let us suppose that

$$\limsup_{n=\infty} \frac{\log |A_n|}{\lambda_n} = a < \infty.$$

If $\epsilon > 0$, then for $n > n_\epsilon$,

$$\frac{\log |A_n|}{\lambda_n} < a + \frac{\epsilon}{2}, \text{ if } a > -\infty, \quad \frac{\log |A_n|}{\lambda_n} < \frac{\epsilon}{2}, \text{ if } a = -\infty.$$

Hence:

$$|A_n| < e^{\lambda_n\left(a+\frac{\epsilon}{2}\right)} \text{ or } |A_n| < e^{\lambda_n \frac{\epsilon}{2}} \text{ (if } a = -\infty). \tag{11}$$

Let us set $a' = \max(0, a)$.

We may write for $q > p \geqq 2$:

$$\sum_p^q a_n e^{-\lambda_n(a'+\epsilon)} = \sum_p^q (A_n - A_{n-1}) e^{-\lambda_n(a'+\epsilon)}$$

$$= \sum_p^{q-1} A_n \left(e^{-\lambda_n(a'+\epsilon)} - e^{-\lambda_{n+1}(a'+\epsilon)}\right) + A_q e^{-\lambda_q(a'+\epsilon)} - A_{p-1} e^{-\lambda_p(a'+\epsilon)}.$$

It follows then from (11) that if $p > 1 + n_\epsilon$ then:

$$\left|\sum_p^q a_n e^{-\lambda_n(a'+\epsilon)}\right| \leqq \sum_p^{q-1} e^{\lambda_n\left(a'+\frac{\epsilon}{2}\right)} \left(e^{-\lambda_n(a'+\epsilon)} - e^{-\lambda_{n+1}(a'+\epsilon)}\right)$$

$$+ e^{\lambda_q\left(a'+\frac{\epsilon}{2}\right)} e^{-\lambda_q(a'+\epsilon)} + e^{\lambda_{p-1}\left(a'+\frac{\epsilon}{2}\right)} e^{-\lambda_p(a'+\epsilon)}$$

$$(12) \leqq \sum_{p}^{q-1} e^{\lambda_n\left(a'+\frac{\epsilon}{2}\right)}(a'+\epsilon)\int_{\lambda_n}^{\lambda_{n+1}} e^{-u(a'+\epsilon)}\,du + e^{-\lambda_q\frac{\epsilon}{2}} + e^{-\lambda_p\frac{\epsilon}{2}}$$

$$\leqq (a'+\epsilon)\sum_{p}^{q-1}\int_{\lambda_n}^{\lambda_{n+1}} e^{-\frac{u\epsilon}{2}}\,du + e^{-\lambda_q\frac{\epsilon}{2}} + e^{-\lambda_p\frac{\epsilon}{2}}$$

$$= (a'+\epsilon)\int_{\lambda_p}^{\lambda_q} e^{-\frac{u\epsilon}{2}}\,du + e^{-\lambda_q\frac{\epsilon}{2}} + e^{-\lambda_p\frac{\epsilon}{2}},$$

and since

$$\int_0^{\infty} e^{-\frac{u\epsilon}{2}}\,du < \infty,$$

one sees immediately that the expression

$$(a'+\epsilon)\int_{\lambda_p}^{\lambda_q} e^{-\frac{u\epsilon}{2}}\,du + e^{-\lambda_q\frac{\epsilon}{2}} + e^{-\lambda_p\frac{\epsilon}{2}}$$

tends to zero as p tends to infinity. This proves that the series (1) converges for $s = a'+\epsilon$, thus that $\sigma_C \leqq a'$.

We shall now prove that if (1) converges for $s = \sigma_0 > 0$, then $a \leqq \sigma_0$. This will prove that if $a = \infty$, then $\sigma_C = \infty$, and that if $0 < a < \infty$, then $a \geqq \sigma_C$. This together with $\sigma_C \leqq a'$ proves the theorem.

We can write

$$A_n = \sum_{1}^{n} a_m = \sum_{2}^{n-1} A_m(\sigma_0)(e^{\lambda_m\sigma_0} - e^{\lambda_{m+1}\sigma_0}) + A_n(\sigma_0)e^{\lambda_n\sigma_0} + a_1 e^{-\lambda_1\sigma_0}(e^{\lambda_1\sigma_0} - e^{\lambda_2\sigma_0}).$$

If $\sum a_n e^{-\lambda_n\sigma_0}$ converges, there exists a constant M such that $|A_n(\sigma_0)| < M$, $(n \geqq 1)$, and

$$|A_n| \leqq M\left[\sum_{1}^{n-1}(e^{\lambda_{m+1}\sigma_0} - e^{\lambda_m\sigma_0}) + e^{\lambda_n\sigma_0}\right]$$

$$= M(2e^{-\lambda_n\sigma_0} - e^{\lambda_1\sigma_0}) < 2Me^{\lambda_n\sigma_0},$$

which proves that

$$a = \limsup_{n=\infty} \frac{\log|A_n|}{\lambda_n} \leqq \sigma_0.$$

The next theorem is an immediate corollary of Theorem V.

THEOREM VI. *If* $\sum a_n e^{-\lambda_n \sigma_0}$ *does not converge, then*

$$\sigma_C = \sigma_0 + \limsup_{n=\infty} \frac{\log \left| \sum_1^n a_m e^{-\lambda_m s_0} \right|}{\lambda_n}.$$

If $\sum |a_n| e^{-\lambda_n \sigma_0} = \infty$, *then:*

$$\sigma_A = \sigma_0 + \limsup_{n=\infty} \frac{\log \left(\sum_1^n |a_m| e^{-\lambda_m \sigma_0} \right)}{\lambda_n}.$$

It should be remarked that in Theorem V the hypothesis $a>0$ (as well as $b>0$) is essential. Suppose indeed that the series (2) represents a function $f(z)$ holomorphic in $|z|<R_1$ with $R_1>1$, and such that $f(1)\neq 0$. The series

$$\varphi(z) = \frac{1}{1-z} f(z) = \sum A_n z^n, \quad (A_n = \sum_1^n a_m),$$

has then the radius of convergence equal to unity, since the point $z=1$ is singular for $\varphi(z)$, and

$$\limsup_{n=\infty} \frac{\log |A_n|}{n} = 0,$$

but, on the other hand, for the series (3), $\sigma_C \leqq -\log R_1 < 0$.

We shall have also to use the following theorem which furnishes cases in which the abscissas of convergence are obtained by formulas analogous to those for Taylor-D series.

THEOREM VII. *If*

$$\limsup_{n=\infty} \frac{n}{\lambda_n} < \infty,$$

then

$$\sigma_A = \sigma_C = \limsup_{n=\infty} \frac{\log |a_n|}{\lambda_n}.$$

Let us suppose that

$$\frac{n}{\lambda_n} < \frac{1}{L} < \infty \ (n \geqq 2),$$

and let us first suppose that $a = \limsup \dfrac{\log |a_n|}{\lambda_n} < \infty$. If $b_1 > a$, we have, for $n > n_{b_1}$: $|a_n| < e^{\lambda_n b_1}$, and for $\sigma > b_1$:

$$|a_n| e^{-\lambda_n \sigma} \leqq e^{\lambda_n (b_1 - \sigma)} < e^{L(b_1 - \sigma) n}.$$

Thus $\sum |a_n| e^{-\lambda_n \sigma} < \infty$ for $\sigma > a$. On the other hand, if $-\infty < a \leqq \infty$, and if $-\infty < \sigma < b_1 < a$, we have for an infinity of n: $|a_n| > e^{\lambda_n b_1}$, and $|a_n| e^{-\lambda_n \sigma} \geqq e^{\lambda_n (b_1 - \sigma)} > 1 (n > 1)$, the series (1) does not converge therefore for $\sigma < a$. This proves the theorem.

Theorem VII holds also, as G. Valiron has proved, if the condition $\limsup \frac{n}{\lambda_n} < \infty$ is replaced by the less restrictive condition $\lim \frac{\log n}{\lambda_n} = 0$ (see [3]).

A Dirichlet series may converge at every point and converge absolutely at no point, that is to say, the case may occur in which $\sigma_C = -\infty$, $\sigma_A = \infty$.

Let us set: $a_n = \frac{(-1)^n}{n}$, $\lambda_n = \sqrt{\log \log n}$, $(n \geqq 3)$.

If $k > 0$, the following relationships hold:

(α) $\lim_{n = \infty} a_n e^{\lambda_n k} = 0$

(β) $|a_n| e^{\lambda_n k} < |a_{n-1}| e^{\lambda_{n-1} k}$ $(n > n_k)$.

The first relationship is obvious. The second can be written in the form:

$$-\log \left(1 - \frac{1}{n}\right) > k \left(\sqrt{\log \log n} - \sqrt{\log \log (n-1)}\right), \quad (n > n_k),$$

which follows from $-\log \left(\frac{n-1}{n}\right) \sim \frac{1}{n}$ and from the relationship

$$\sqrt{\log \log n} - \sqrt{\log \log (n-1)} \sim \frac{1}{2n \log n \sqrt{\log \log n}}.$$

The series $\sum a_n e^{\lambda_n k}$ (alternating, for n sufficiently large) converges therefore for $k > 0$ arbitrary. This proves that $\sigma_C = -\infty$. On the other hand, $\sum_1^n |a_m| \sim \log n$, hence

$$\sigma_A = \limsup \frac{\log \sum_1^n |a_m|}{\lambda_n} = \lim \frac{\log \log n}{\sqrt{\log \log n}} = \infty.$$

In this case $\limsup \frac{\log n}{\lambda_n} = \infty$. Let us remark that in this example the quantity a of Theorem V is equal to zero.

The following theorem gives a general relationship between the two abscissas of convergence.

THEOREM VIII. *For every Dirichlet series*

$$\sigma_A - \sigma_C \leqq \limsup_{n=\infty} \frac{\log n}{\lambda_n}. \tag{13}$$

The examples above show that the sign of equality cannot be omitted in this theorem.

Let us set

$$\limsup_{n=\infty} \frac{\log n}{\lambda_n} = b.$$

Any positive ϵ being given, we have to prove that

$$\sum |a_n| e^{-\lambda_n(\sigma_c+b+\epsilon)} < \infty.$$

From the convergence of (1) for $s = \sigma_C + \frac{\epsilon}{2}$ it follows that

$$|a_n| e^{-\lambda_n\left(\sigma_c+\frac{\epsilon}{2}\right)} < A < \infty,$$

and

$$|a_n| e^{-\lambda_n(\sigma_c+b+\epsilon)} < A e^{-\lambda_n\left(b+\frac{\epsilon}{2}\right)};$$

but for $n > n_\epsilon$:

$$\frac{\log n}{\lambda_n} < b + \frac{\epsilon}{4}.$$

Hence

$$|a_n| e^{-\lambda_n(\sigma_c+b+\epsilon)} < A e^{-\log n \left(\frac{b+\frac{\epsilon}{2}}{b+\frac{\epsilon}{4}}\right)} = A n^{-\frac{b+\frac{\epsilon}{2}}{b+\frac{\epsilon}{4}}},$$

which proves the theorem.

If $\sigma_C < \infty$, the series (1) converges uniformly in every closed bounded region situated in the half-plane $\sigma > \sigma_C$. This series represents then an analytic function $f(s)$ holomorphic for $\sigma > \sigma_C$:

$$f(s) = \sum_1^\infty a_n e^{-\lambda_n s}. \tag{14}$$

By $f(s)$ we shall understand not only the sum of the series, but generally the function given by its analytic continuation, and we shall call it *the function represented by the series* $\sum a_n e^{-\lambda_n s}$. The actual sum of this series in $\sigma > \sigma_C$ will be called *the principal branch of* $f(s)$ *in* $\sigma > \sigma_C$. To be more specific, the corresponding abscissas σ_C, σ_A will be denoted, when needed, by σ_C^f, σ_A^f.

It is obvious that the principal branch of $f(s)$ is bounded in $\sigma \geqq \sigma_A + \epsilon$, if $\sigma_A > -\infty$, $(\sigma_A < \infty)$, the quantity $\epsilon > 0$ being chosen arbitrarily. We have as a matter of fact, in this half-plane:

$$|f(s)| \leqq \sum |a_n| e^{-\lambda_n(\sigma_A + \epsilon)}.$$

Before we close this chapter we should remark that in contrast to a Taylor-*D* series, a general Dirichlet series does not converge *uniformly* in every half-plane $\sigma \geqq \sigma_0 > \sigma_C (\sigma_C < \infty)$, although it does converge uniformly in every *bounded* closed region of such a half-plane. This is, for instance, the case of the series $\sum \frac{(-1)^n}{n^s}$, which does not converge uniformly in the half-plane $\sigma \geqq \sigma_0 > 0$ with $\sigma_0 < 1$, and yet here $\sigma_C = 0$. A Dirichlet series does converge uniformly, as is readily seen, in each half-plane $\sigma > \sigma_1 > \sigma_A (\sigma_A < \infty)$. H. Bohr introduced the notion of the abscissa σ_u of uniform convergence which is the g. l. b. of quantities σ_0, such that (1) converges uniformly for $\sigma \geqq \sigma_0$. Obviously $\sigma_C \leqq \sigma_u \leqq \sigma_A$. Bohr has shown (see [3]) that

$$\sigma_u = \limsup_{n=\infty} \frac{\log\left(\text{l. u. b.}_{-\infty < t < \infty} \left| \sum_1^n a_m e^{i\lambda_m t} \right|\right)}{\lambda_n},$$

if the right-hand expression is positive.

II

EVALUATION OF THE COEFFICIENTS

We shall extend to Dirichlet series the well known Cauchy formula for the coefficients of a Taylor series. We shall suppose that $\sigma_A < \infty$, although this condition is not essential. But we shall not have to use the most general cases in which the formulas, established here with this restrictive hypothesis, are still true.

THEOREM IX. *If*

$$f(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$$

with $\sigma_A^f < \infty$, *then*

$$a_n e^{-\lambda_n \sigma_1} = \lim_{T=\infty} \frac{1}{T} \int_{t_0}^{T} f(\sigma_1 + it) e^{\lambda_n it} dt \quad (n \geqq 1), \tag{15}$$

where t_0 *is arbitrary, where* $\sigma_1 > \sigma_A^f$, *otherwise arbitrary, and where* $f(\sigma_1+it)$ *is the value of the principal branch of the function.*

We have with $s = \sigma_1 + it$:

$$\frac{1}{T}\int_{t_0}^{T} f(s) e^{\lambda_n s} dt = \frac{1}{T}\int_{t_0}^{T} \Big(\sum_{1}^{\infty} a_m e^{(\lambda_n - \lambda_m)s}\Big) dt$$

$$= \frac{1}{T}\int_{t_0}^{T} \Big(\sum_{1}^{n-1} a_m e^{(\lambda_n - \lambda_m)s}\Big) dt + \frac{1}{T}\int_{t_0}^{T} a_n dt + \frac{1}{T}\int_{t_0}^{T} \Big(\sum_{n+1}^{\infty} a_m e^{(\lambda_n - \lambda_m)s}\Big) dt.$$

Since, for k real, $k \neq 0$:

$$\lim_{T=\infty} \frac{1}{T}\int_{t_0}^{T} e^{k(\sigma_1 + it)} dt = 0,$$

and since the series under the integral signs converge uniformly with respect to $t(-\infty < t < \infty)$, we see that:

$$\lim_{T=\infty} \frac{1}{T}\int_{t_0}^{T} \Big(\sum_{1}^{n-1} a_m e^{(\lambda_n - \lambda_m)s}\Big) dt = 0$$

$$\lim_{T=\infty} \int_{t_0}^{T} \left(\sum_{n+1}^{\infty} a_m e^{(\lambda_n - \lambda_m)s} \right) dt = 0.$$

Hence

$$\lim_{T=\infty} \frac{1}{T} \int_{t_0}^{T} f(s) e^{\lambda_n s} dt = \lim_{T=\infty} \frac{1}{T} \int_{t_0}^{T} a_n dt = a_n,$$

which is equivalent to the statement of the theorem.

As a matter of fact, it is seen that in Theorem IX, $\sigma_A{}^f$ can be replaced by $\sigma_u{}^f$, since only the uniform convergence was used in the proof.

We shall have to use the following lemma.

LEMMA I. *If $c>0$, if k is a positive integer, and if ω is real, then:*

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\omega(c+it)}}{(c+it)^k} dt \Bigg\} = \frac{\omega^{k-1}}{(k-1)!}, \quad (\omega > 0) \qquad (16)$$
$$= 0, \quad (\omega \leqq 0)$$

Let us first suppose $\omega>0$. If $T>0$, $\sigma_1<0$, we shall denote by I_1 $(\pm T, \sigma_1)$ the segments $(\sigma_1 \leqq \sigma \leqq c, t = \pm T)$ and by $I_2(T,\sigma_1)$, $I_3(T)$ the segments $(\sigma = \sigma_1, |t| \leqq T)$ and $(\sigma = c, |t| \leqq T)$. By $C(T,\sigma_1)$ we shall denote the rectangle composed of the four segments. We have by the theorem on residues:

$$\frac{1}{2\pi i} \oint_{C(T,\sigma_1)} \frac{e^{\omega s}}{s^k} ds = \frac{\omega^{k-1}}{(k-1)!}. \qquad (17)$$

Since for σ_1 fixed the integrals extended over $I_1(T,\sigma_1)$ and $I_1(-T,\sigma_1)$ tend to zero, we see that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\omega(c+it)}}{(c+it)^k} dt - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{\omega(\sigma_1+it)}}{(\sigma_1+it)^k} dt = \frac{\omega^{k-1}}{(k-1)!},$$

and since the second integral in this equality tends to zero as σ_1 tends to $-\infty$, we see that our equality holds for $\omega>0$. The proof is similar if $\omega \leqq 0$. Here $C(T,\sigma_1)$ should be replaced by a contour $C'(T,\sigma_2)$ with $\sigma_2>c$ and composed of segments I_1' $(\pm T,\sigma_2) \equiv (c \leqq \sigma \leqq \sigma_2, t = \pm T)$, $I_2(T,\sigma_2) \equiv (\sigma = \sigma_2,$

$|t| \leqq T$), $I_3(T)$. The integral (17), in which $C(T,\sigma_1)$ is replaced by $C'(T,\sigma_2)$, is then zero, since the function under the integral sign is holomorphic inside $C'(T,\sigma_2)$, and on making first T tend to $+\infty$ and then σ_2 to $+\infty$ we obtain the desired formula.

THEOREM X. *If*

$$f(s) = \sum a_n e^{-\lambda_n s}$$

with $\sigma_A{}^f < \infty$, *if* $\nu \geqq 0$ *and if* k *is a positive integer, then*

$$\left.\begin{array}{ll} \text{for } \nu > \lambda_1\text{:} & \sum_{\lambda_n < \nu} (\nu - \lambda_n)^{k-1} a_n \\ \text{for } 0 \leqq \nu \leqq \lambda_1\text{:} & 0 \end{array}\right\} = \frac{(k-1)!}{2\pi} \int_{-\infty}^{\infty} \frac{f(c+it)e^{\nu(c+it)}}{(c+it)^k} dt$$

where $c > \max(\sigma_A{}^f, 0)$.

We have, with $s = c+it$ for $\nu > \lambda_1$:

$$\int_{-\infty}^{\infty} \frac{f(s)e^{\nu s}}{s^k} dt = \int_{-\infty}^{\infty} \frac{(\sum a_n e^{-\lambda_n s})e^{\nu s} dt}{s^k}$$

$$= \int_{-\infty}^{\infty} \frac{\sum_{\lambda_n < \nu} a_n e^{(\nu-\lambda_n)s}}{s^k} dt + \int_{-\infty}^{\infty} \frac{\sum_{\lambda_n \geqq \nu} a_n e^{(\nu-\lambda_n)s}}{s^k} dt.$$

From Lemma I and from the uniform convergence of the series under the last integral sign we see that the last integral is zero. Therefore by Lemma I:

$$\frac{(k-1)!}{2\pi} \int_{-\infty}^{\infty} \frac{f(s)}{s^k} e^{\nu s} dt = \frac{(k-1)!}{2\pi} \int_{-\infty}^{\infty} \left(\sum_{\lambda_n < \nu} a_n e^{(\nu-\lambda_n)s} \right) \frac{dt}{s^k}$$

$$= \frac{(k-1)!}{2\pi} \sum_{\lambda_n < \nu} a_n \int_{-\infty}^{\infty} \frac{e^{(\nu-\lambda_n)s}}{s^k} dt = \sum_{\lambda_n < \nu} a_n (\nu - \lambda_n)^{k-1}.$$

If $\nu \leqq \lambda_1$ then

$$\int_{-\infty}^{\infty} \frac{f(s)e^{\nu s}}{s^k} dt = \int_{-\infty}^{\infty} \left(\sum_{\lambda_1 \geqq \nu} a_n e^{(\nu-\lambda_n)s} \right) \frac{dt}{s^k} = 0.$$

In this theorem, too, $\sigma_A{}^f$ can be replaced by $\sigma_u{}^f$.

III

ARITHMETICAL PROPERTIES OF THE SEQUENCE OF EXPONENTS

The properties of the function $f(s)$ represented by the series (1) depend on both sequences $\{\lambda_n\}$ and $\{a_n\}$. The properties of $\{\lambda_n\}$ which are mostly involved in the modern theory of Dirichlet series, are of two different kinds: a) the arithmetical character of the sequence; b) the growth of the number $N(t)$ of quantities λ_n smaller than t as t tends to infinity.

We shall begin with some theorems on Dirichlet series in which the λ_n have simple arithmetical properties.

Let us begin by studying a certain analytical transformation.

In the z-plane we define the region $\Delta(\alpha)$, $(\alpha>0)$ as follows: the frontier $C(\alpha)$ of $\Delta(\alpha)$ is composed of the segments

$$\left.\begin{matrix} y=(2k+1)\pi \\ -\alpha \leqq x \leqq 0 \end{matrix}\right\} (k=0, \pm 1, \pm 2, \cdots ; z=x+yi), \tag{18}$$

and the curve

$$x=-\alpha \sin^2\left(\frac{y}{2}\right), \tag{19}$$

$\Delta(\alpha)$ containing the positive real axis. $\Delta(\alpha)$ contains therefore also the points with $x>0, y=(2k+1)\pi$, and all the points with $y \neq (2k+1)\pi$, $x>-\alpha \sin^2\left(\frac{y}{2}\right)$.

In the s-plane, $(s=\sigma+it)$, we define the region $D(\lambda)$ $\left(0<\lambda<\frac{1}{2}\right)$ as the region containing the real positive axis of which the frontier is the curve $L(\lambda)$ given by

$$\sigma = \lambda \sin^2\left(\frac{t - \left[\frac{t}{2\pi}\right]2\pi}{4}\right), \tag{20}$$

where $[a]$ denotes the integer nearest to a (if $\frac{t}{2\pi} = n + \frac{1}{2}$, where n is an integer, $\left[\frac{t}{2\pi}\right]$ can be taken equal as well to n as to $n+1$, since the value of σ in (20) remains then the same). Each point of $L(\lambda)$ is clearly at a distance not greater than $\frac{\lambda}{2}$ from the t-axis; only the points $\left(\frac{\lambda}{2}, (2n+1)\pi\right)$ of $L(\lambda)$ are at the distance $\frac{\lambda}{2}$ from this axis.

Let us set, for $z \in \Delta(\alpha)$:

$$s = \varphi(z) = z - \log(1 + e^{-z}) + \log 2, \tag{21}$$

with $\log(1+e^{-x}) > 0$, $(x > 0)$, the function $\log(1+e^{-z})$ being defined for other points of $\Delta(\alpha)$ by continuity.

Lemma II. *If* $0 < \lambda < \frac{1}{2}$, *there exists a positive* α *such that if* $z \in \Delta(\alpha)$, *then* $s = \varphi(z) \in D(\lambda)$.

We prove first that the points satisfying (19) are mapped on $\overline{D(\lambda)}$. We have:

$$\begin{aligned} |1 + e^{\alpha \sin^2\left(\frac{y}{2}\right) - iy}|^2 &= 1 + 2e^{\alpha \sin^2\left(\frac{y}{2}\right)} \cos y + e^{2\alpha \sin^2\left(\frac{y}{2}\right)} \\ &= 1 + \sum_0^\infty \frac{\left(2\alpha \sin^2\left(\frac{y}{2}\right)\right)^n}{n!} + 2\cos y \sum_0^\infty \frac{\left(\alpha \sin^2\left(\frac{y}{2}\right)\right)^n}{n!} \\ &= 2(1 + \cos y) + \alpha \sin^2\left(\frac{y}{2}\right) . A(\alpha, y), \end{aligned}$$

where $A(\alpha, y)$ is given by a series which converges uniformly when $0 < \alpha \leqq \alpha_1$, $-\infty < y < \infty$, $\alpha_1 > 0$ being any fixed quantity; and thus, for $0 < \alpha < \alpha_1$, $|A(\alpha, y)| < M < \infty$. Hence, to every $\epsilon > 0$ there corresponds an $\alpha(\epsilon)$ such that for $0 < \alpha < \alpha(\epsilon)$:

$$|1+e^{\alpha \sin^2\left(\frac{y}{2}\right)-iy}|^2 \leqq 4-4\sin^2\left(\frac{y}{2}\right)+M\alpha\sin^2\left(\frac{y}{2}\right)$$
$$\leqq 4-4(1-\epsilon)\sin^2\left(\frac{y}{2}\right).$$

Hence if $\lambda<\lambda'<\frac{1}{2}$, we have for α sufficiently small:

$$|1+e^{\alpha \sin^2\left(\frac{y}{2}\right)-iy}|^2 \leqq 4\left(1-2\lambda'\sin^2\left(\frac{y}{2}\right)\right),$$

and

$$\log|1+e^{\alpha \sin^2\left(\frac{y}{2}\right)-iy}| \leqq \log 2+\frac{1}{2}\log\left(1-2\lambda'\sin^2\left(\frac{y}{2}\right)\right) \tag{22}$$
$$\leqq \log 2-\lambda'\sin^2\left(\frac{y}{2}\right).$$

If now z satisfies (19) we have for $s=\sigma+it=\varphi(z)$:

$$\sigma=\sigma(y)=-\alpha\sin^2\left(\frac{y}{2}\right)-\log|1+e^{\alpha \sin^2\left(\frac{y}{2}\right)-iy}|+\log 2,$$

and if α is sufficiently small, we have by (22):

$$\sigma=\sigma(y)\geqq -\alpha\sin^2\left(\frac{y}{2}\right)+\lambda'\sin^2\left(\frac{y}{2}\right)\geqq\lambda\sin^2\left(\frac{y}{2}\right). \tag{23}$$

On the other hand, we have for all y:

$$\langle \operatorname{Arg}(1+e^{\alpha \sin^2\left(\frac{y}{2}\right)-iy})\rangle \leqq \langle y\rangle, \tag{24}$$

where $\langle a\rangle=|a-\left[\frac{a}{2\pi}\right]2\pi|$. And, since, if $s=\varphi(z)=\varphi(x+iy)$,

$$t=y-\operatorname{Arg}(1+e^{-z})$$
$$\langle t\rangle\leqq\langle y\rangle+\langle\operatorname{Arg}(1+e^{-z})\rangle,$$

we see by (24) that $\langle t\rangle\leqq 2\langle y\rangle$, that is to say,

$$|t-\left[\frac{t}{2\pi}\right]2\pi|\leqq 2|y-\left[\frac{y}{2\pi}\right]2\pi|,$$

and

$$\sin^2\left(\frac{y}{2}\right)=\sin^2\left(\frac{\langle y\rangle}{2}\right)\geqq\sin^2\left(\frac{\langle t\rangle}{4}\right)=\sin^2\left(\frac{t-\left[\frac{t}{2\pi}\right]2\pi}{4}\right).$$

It follows now from (23) that if z satisfied (19) and if $s=\varphi(z)$ then:

$$\sigma \geqq \lambda \sin^2\left(\frac{t-\left[\frac{t}{2\pi}\right]2\pi}{4}\right), \tag{25}$$

that is to say, if z satisfies (19) with $\alpha>0$ sufficiently small: $\alpha \leqq \alpha_0$, then $s=\varphi(z) \in \overline{D(\lambda)}$.

Let us remark that $|1+e^{\alpha \sin^2\left(\frac{y}{2}\right)-iy}|$ (with y fixed), decreases as α decreases to $-\infty$. Therefore (22) and (23) hold also if z satisfies (19) with $-\infty<\alpha \leqq \alpha_0$. Since (24) is true for each fixed value of α, we see that (25) with the sign $>$ is true for $s=\varphi(z)$ where z is any point of $\Delta(\alpha_0)$ (open). We have thus proved that if $z \in \Delta(\alpha_0)$, then $s \in D(\lambda)$, which is the statement which was to be proved.

We now prove the following theorem [8]:

THEOREM XI. *If $f(s)$ is holomorphic and bounded in a region $D(\lambda)$ with $0<\lambda<\frac{1}{2}$, if for σ sufficiently large*

$$f(s)=\sum_1^\infty a_n e^{-\lambda_n s},$$

where the sequence $\{\lambda_n\}$ is such that for no couple of two integers n, $m(n\neq m)$ is $\lambda_n-\lambda_m$ an integer and if $\sigma_A^f<\infty$, then the sequence $\{a_n\}$ is bounded.

If we write $s=\varphi(z)=z-\log(1+e^{-z})+\log 2$, we get from the relationship:

$$(1+e^{-z})^{\lambda_n}=\sum_0^\infty T_n^{(m)} e^{-mz},$$

where $\mathcal{R}(z)=x>0$, and where:

$$T_n^{(m)}=\frac{\lambda_n(\lambda_n-1)\cdots(\lambda_n-m+1)}{m!},$$

the equality:

$$\sum_n a_n e^{-\lambda_n s}=\sum_n \frac{a_n}{2^{\lambda_n}} e^{-\lambda_n z}(1+e^{-z})^{\lambda_n} \tag{26}$$

$$=\sum_n \frac{a_n}{2^{\lambda_n}} e^{-\lambda_n z}\sum_m T_n^{(m)} e^{-mz}=\sum_{r=1}^\infty b_r e^{-\mu_r z}.$$

Here $\{\mu_n\}$ is a sequence of positive numbers tending to infinity and is such that to each r there corresponds a positive integer n and a non-negative integer m such that $\mu_r=\lambda_n+m$. Since we suppose that no two λ_n differ by an integer, to each r there corresponds only one such couple (n,m). We shall show that the last series in (26) converges absolutely for x sufficiently large, which means that the double series in this equality converges absolutely, since

$$b_r e^{-\mu_r x}=\frac{a_n}{2^{\lambda_n}}T_n^{(m)}e^{-(\lambda_n+m)x}, \tag{27}$$

where n and m are the two unique integers which correspond to r.

Since

$$\left|\frac{T_n^{(m)}}{T_n^{(m-1)}}\right|=\left|\frac{\lambda_r-m+1}{m}\right|\geqq 1,$$

if, and only if, $m\leqq\frac{\lambda_n+1}{2}$, we see that:

$$\max_m|T_n^{(m)}|=T_n^{\left(E\left(\frac{\lambda_n+1}{2}\right)\right)}, \tag{28}$$

where $E(c)$ denotes the largest integer not larger than c.

It follows from (28) that, on setting

$$\alpha_n=E(\lambda_n),\ \beta_n=E\left(\frac{\lambda_n+1}{2}\right):$$

$$C_{\alpha_n}^{\beta_n}\leqq\max_m|T_n^{(m)}|\leqq C_{\alpha_n+1}^{\beta_n} \tag{29}$$

(the left-hand inequality having a meaning when $\lambda_n\geqq 1$). But if α, β, with $\alpha>\beta$, are two integers tending to infinity, then

$$C_\alpha^\beta=\frac{\alpha!}{\beta!(\alpha-\beta)!}\sim\frac{\sqrt{\alpha}}{\sqrt{2\pi}\sqrt{\beta(\alpha-\beta)}}\cdot\frac{\alpha^\alpha}{\beta^\beta(\alpha-\beta)^{\alpha-\beta}},$$

and if $\alpha=2\beta+O(1)$, $\beta\to\infty$, then

$$\log\frac{\alpha^\alpha}{\beta^\beta(\alpha-\beta)^{\alpha-\beta}}-\alpha\log 2=O(1).$$

Thus there exist two positive constants M_1, M_2 such that

$$\frac{M_1 2^{\lambda_n}}{\sqrt{\lambda_n}} < \max_m |T_n^{(m)}| < \frac{M_2 2^{\lambda_n}}{\sqrt{\lambda_n}}. \tag{30}$$

It follows then from (27) and (28) that, if $r=r(n)$ is such that $\mu_r=\lambda_n+E\left(\frac{\lambda_n+1}{2}\right)$, then

$$|a_n| = \frac{2^{\lambda_n}}{\max_m |T_n^{(m)}|}\,|b_{r(n)}| \leqq \frac{1}{M_1}\sqrt{\lambda_n}\,|b_{r(n)}|. \tag{31}$$

Moreover by (26):

$$\sum_{r=1}^{\infty} |b_r|\, e^{-\mu_r x} \leqq \sum_n \frac{|a_n|}{2^{\lambda_n}} e^{-\lambda_n x} \sum_m T_n^{(m)} e^{-mx}$$

$$\leqq M_2 \sum_n \frac{|a_n|}{\sqrt{\lambda_n}} e^{-\lambda_n x} \sum_m e^{-mx} = M_2 \sum \frac{|a_n|}{\sqrt{\lambda_n}} \frac{e^{-\lambda_n x}}{1-e^{-x}}.$$

The last series converges for $x > \max(\sigma_A^f, 0)$, hence $\sum b_r e^{-\mu_r z}$ has an abscissa of absolute convergence ($<\infty$). By Theorem IX we may therefore write:

$$b_r = \lim_{T=\infty} \frac{1}{T}\int_0^T F(x_1+iy)\, e^{\mu_r(x_1+iy)}\, dy, \tag{32}$$

where $F(z)=\sum b_r e^{-\mu_r z}$, and where x_1 is sufficiently large.

By Lemma II, since $f(s)$ is holomorphic and bounded in some region $D(\lambda)$ with $0<\lambda<\frac{1}{2}$, $F(z)$ is holomorphic and bounded in $\Delta(\alpha)$ where $\alpha>0$ is chosen sufficiently small.

On denoting, for r sufficiently large, by $C_r(\alpha)$ the curve in the z-plane composed of segments

$$\begin{aligned} &y=(2k+1)\pi \\ &-\alpha+\frac{1}{\mu_r} \leqq x \leqq 0, \end{aligned} \tag{33}$$

and the curve

$$x = -\alpha \sin^2\left(\frac{y}{2}\right)+\frac{1}{\mu_r}, \tag{34}$$

we see, by an immediate application of Cauchy's theorem, that from (32) follows the equality

$$b_r = \lim_{T=\infty} \frac{1}{Ti} \int_{C_r(\alpha,T)} F(z) e^{\mu_r z}\, dz, \tag{35}$$

where $C_r(\alpha, T)$ is the part of the curve $C_r(\alpha)$ which begins at the point $\left(\frac{1}{\mu_r}, 0\right)$ and ends at the point with ordinate T, the segments of form (33) with $0 < (2k+1)\pi < T$, which therefore belong to $C_r(\alpha, T)$, being counted twice, first running from $\left(-\alpha+\frac{1}{\mu_r}, (2k+1)\pi\right)$ to $(0, (2k+1)\pi)$ and afterwards in the opposite sense (the direction on $C_r(\alpha, T)$ is defined by the fact that it begins at $\left(\frac{1}{\mu_r}, 0\right)$). Formula (35) follows from the fact that the integrals of $F(z)e^{\mu_r z}$ over the two segments $\left(\frac{1}{\mu_r} \leqq x \leqq x_1, t=0\right)$ and the segment with $t=T$ situated between $C_r(\alpha)$ and the line $x=x_1$ are bounded.[1]

It follows then from (35) that

$$|b_r| \leqq M\left[\int_0^{\pi} e^{\left(-\alpha \sin^2\left(\frac{y}{2}\right)+\frac{1}{\mu_r}\right)\mu_r}\, dy + \int_{-\alpha+\frac{1}{\mu_r}}^{0} e^{\mu_r x}\, dx\right],$$

where M is a positive constant. We have thus:

$$|b_r| \leqq M_1 \int_0^{\pi} e^{-\alpha\mu_r \sin^2\left(\frac{y}{2}\right)}\, dy + \frac{M}{\mu_r}(1-e^{1-\alpha\mu_r}). \tag{36}$$

For every fixed r sufficiently large, we choose η such that $\frac{1}{\mu_r} < \alpha^2 \sin^2 \frac{\eta}{2} < \frac{2}{\mu_r}$, and write

[1] On the two sides of the segments of form (33) $F(z)$ is defined by continuity from its values in $\Delta(\alpha)$. It is obvious that, for $z=(2k+1)\pi i$, $F(z)$ should be taken equal to $\lim_{\sigma=\infty} f(s) = a_1 \lim_{\sigma=\infty} e^{-\lambda_1 s} (=0$ if $\lambda_1 > 0$, $= a_1$ if $\lambda_1 = 0)$. This definition by continuity is possible since if z in $\Delta(\alpha)$ tends to a point of a segment (33) then $s=\varphi(z)$ tends either to a point of $\Delta(\lambda)$ or to $+\infty$.

$$\int_0^{\pi} e^{-\alpha\mu_r \sin^2\left(\frac{y}{2}\right)} dy < 2\left(\int_0^{\eta} e^{-\alpha\mu_r \sin^2\left(\frac{y}{2}\right)} dy + \int_{\eta}^{\frac{\pi}{2}} e^{-\alpha\mu_r \sin^2\left(\frac{y}{2}\right)} dy\right).$$

The first of the integrals on the right hand is smaller than η. For the second integral we have the immediate estimate:

$$\int_{\eta}^{\frac{\pi}{2}} e^{-\alpha\mu_r \sin^2\left(\frac{y}{2}\right)} dy < \frac{\sqrt{2}}{\sin\frac{\eta}{2}} \int_0^{\frac{\pi}{2}} e^{-\alpha\mu_r \sin^2\left(\frac{y}{2}\right)} d\left(\sin^2\left(\frac{y}{2}\right)\right)$$

$$= \frac{\sqrt{2}}{\sin\frac{\eta}{2}} \int_0^{\frac{1}{2}} e^{-\alpha\mu_r u} du = O\left(\frac{1}{\sqrt{\mu_r}}\right).$$

It follows thus from (36) that

$$|b_r| = O\left(\frac{1}{\sqrt{\mu_r}}\right),$$

and if $r = r(n)$, that is to say, if $\mu_r = \lambda_n + E\left(\frac{\lambda_n + 1}{2}\right)$, then

$$|b_{r(n)}| = O\left(\frac{1}{\sqrt{\lambda_n}}\right).$$

From (31) it follows then that $|a_n|$ is bounded ($n \geqq 1$) and our theorem is proved.

We shall now recall a definition concerning sequences. The elements of a sequence $\{p_n\}$ ($n \geqq 1$) are said to be *linearly independent* if from each equality of the form

$$\sum_1^n A_m p_m = 0,$$

where the A_m are integers, it follows that all these integers are equal to zero.

We are now in a position to prove the following theorem:

THEOREM XII. *If $f(s)$ is holomorphic in a region $D(\lambda)$, with $0 < \lambda < \frac{1}{2}$, and if, for σ sufficiently large,*

$$f(s) = \sum_1^{\infty} a_n e^{-\lambda_n s}$$

where $\sigma_A^f < \infty$, *where the sequence* $\{\lambda_n\}$ *is such that the sequence composed of all the* λ_n *and unity is composed of linearly independent elements, and if there exist a constant* $a \neq 0$ *and a positive quantity* ϵ *such that in* $D(\lambda)$:

$$|f(s) - a| > \epsilon, \tag{37}$$

then $|a| > \epsilon$, *and*

$$|a_n| \leqq |a| - \epsilon, \quad (n \geqq 1). \tag{38}$$

Consider the function $F(s) = \dfrac{1}{a - f(s)}$. If $\sigma > \sigma_A^f$ we have in $D(\lambda)$:

$$|f(s)| < \sum |a_n| e^{-\lambda_n \sigma} = e^{-\lambda_1 \sigma} \sum |a_n| e^{-(\lambda_n - \lambda_1)\sigma},$$

and since, obviously, $\lambda_1 \neq 0$, we see that $\lim_{\sigma = \infty} f(s) = 0$ uniformly with respect to t. This proves that $|a| > \epsilon$, and that the function $F(s)$, which is obviously holomorphic and bounded in $D(\lambda)$, is also given for σ sufficiently large (such that $\sum |a_n| e^{-\lambda_n \sigma} < |a|$) by the equality:

$$F(s) = \frac{1}{a - \sum_n a_n e^{-\lambda_n s}} = \frac{1}{a} \sum_m \left(\frac{\sum_n a_n e^{-\lambda_n s}}{a} \right)^m = \sum_k c_k e^{-L_k s},$$

where the sequence $\{L_k\}$ of non-negative, increasing quantities tends to infinity. Since each L_k is of the form

$$L_k = A_1 \lambda_1 + A_2 \lambda_2 + \cdots + A_m \lambda_m,$$

where the A_i are integers, we see, by the hypothesis on the sequence $\{\lambda_n\}$, that if $k_1 \neq k_2$, $L_{k_1} - L_{k_2}$ is not an integer. By Theorem XI we see then that there exists a constant $M < \infty$, such that $|C_k| < M (k \geqq 0)$. On the other hand, it is readily seen that the sequence $\{L_k\}$ contains all the quantities of the form $m\lambda_n$, where m and n are arbitrary positive integers, and that if $L_k = m\lambda_n$ then $C_k = \dfrac{a_n^m}{a^{m+1}}$. Therefore for n arbitrary, fixed, and for every positive m we have:

$$\frac{|a_n|^m}{|a|^{m+1}} < M,$$

and

$$|a_n| < |a|^{\frac{m+1}{m}} M^{\frac{1}{m}} \ (m \geqq 1).$$

Hence

$$|a_n| \leqq \lim_{m=\infty} \left(|a|^{\frac{m+1}{m}} M^{\frac{1}{m}} \right) = |a|.$$

In this inequality $|a|$ can be replaced by $|a| - \epsilon$, since from $|f(s) - a| > \epsilon$ in $D(\lambda)$ it follows immediately that, if $a = |a| e^{i\varphi}$, then, for each $0 < \theta < 1$,

$$|f(s) - (|a| - \theta\epsilon) e^{i\varphi}| > (1 - \theta)\epsilon$$

in $D(\lambda)$, and therefore, in what precedes $|a|$ can be replaced by $|a| - \theta\epsilon$. That is to say, we have $|a_n| \leqq |a| - \theta\epsilon$, and since θ has only to satisfy the inequality $0 < \theta < 1$, our theorem is proved.

Theorem XII may also be stated as follows:

THEOREM XIII. *If the sequence* $\{\lambda_n\}$ *satisfies the same conditions as in Theorem XII, if* $f(s)$ *is holomorphic in* $D(\lambda)$, $\left(0 < \lambda < \frac{1}{2}\right)$ *and is given for* σ *sufficiently large by* $f(s) = \sum a_n e^{-\lambda_n s}$ *with* $\sigma_A^f < \infty$, *and if*

$$A = \text{l. u. b.}_{n \geqq 1} |a_n|,$$

then the values which $f(s)$ *takes when* $s \in D(\lambda)$ *are everywhere dense in the circle with radius* A *around the origin.*

Theorems XI and XII were proved by the author [**9**]. They bear a character analogous to that of the well known theorem of Weierstrass by which an entire function takes values everywhere dense in the whole plane. A sharper form of Theorem XIII was later given by Aronszajn [**1**,**2**]. This author, on using Theorem XII and the modular function, obtained a theorem which is in character analogous to Picard's theorem on entire (or meromorphic) functions, by which each such function takes all the possible values except at most one finite value (two for meromorphic).

Aronszajn's theorem can be stated in the following manner:

THEOREM XIV. *If the sequence* $\{\lambda_n\}$ *satisfies the same conditions as in Theorem XII, if* $f(s)$ *is holomorphic in* $D(\lambda)$ *with* $0<\lambda<\frac{1}{2}$ *and is given for* σ *sufficiently large by* $\sum a_n e^{-\lambda_n s}$ *with* $\sigma_A{}^f<\infty$, *then* $f(s)$ *takes in* $D(\lambda)$ *all the possible values situated in the circle with radius* $A=\sum_1^\infty |a_n|$ *except at most the value zero and one other value.*

If $a\neq b$ then it is known [**15**] that there exists, in the complex plane a function $\chi(u)$ with the following properties: $\alpha)$ $\chi(u)$ is regular at the origin and is therefore given, in its neighborhood, by $\chi(u)=\sum A_n u^n$, $\beta)$ $\chi(u)$ can be continued analytically on each curve which does not pass through a or b, the points $u=a$, $u=b$ being branch points for $\chi(u)$, $\gamma)$ $|\chi(u)|<1$, $\delta)$ obviously

$$\limsup_{n=\infty} |A_n|^{\frac{1}{n}}=\frac{1}{\min(|a|,|b|)}=\frac{1}{c}. \tag{39}$$

Suppose now that, contrary to the assertion, $f(s)$ does not take two values a, b, such that $a\neq b$, $a\neq 0$, $b\neq 0$, $|a|<A$, $|b|<A$, and let us consider the function $F(s)=\chi[f(s)]$, where $\chi(u)$ is the function defined above. $F(s)$ is holomorphic in $D(\lambda)$. If σ' is chosen such as to have $\sum |a_n|e^{-\lambda_n\sigma'}<c$, we see that $\sum_m |A_m|(\sum |a_n|e^{-\lambda_n\sigma'})^m$ converges, and therefore $F(s)$ is given for σ sufficiently large by a Dirichlet series:

$$F(s)=\sum A_m(f(s))^m$$

$$=\sum_n A_n \sum_{\substack{m_1+\cdots+m_p=n \\ 1\leq l_i<\infty \\ i=1,2,\cdots p}} \frac{n!}{m_1!\cdots m_p!}(a_{l_1})^{m_1}\cdots(a_{l_p})^{m_p}e^{-s(m_1\lambda_{l_1}+\cdots+m_p\lambda_{l_p})}$$

$$=\sum c_k e^{-L_k s},$$

which admits an abscissa of absolute convergence. We have in $D(\lambda)$: $|F(s)|<1$. Here, as in the proof of Theorem XII, we have $L_{k_1}-L_{k_2}\neq$ integer, and, by Theorem XI, we should have $|c_k|<M<\infty\,(k\geqq 1)$. But, if $L_k=m_1\lambda_{l_1}+\cdots+m_p\lambda_{l_p}$,

then $c_k = \frac{n!}{m_1! \cdots m_p!} (a_{l_1})^{m_1} \cdots (a_{l_p})^{m_p}$, and thus:

$$\frac{n!}{m_1! \cdots m_p!} |a_{l_1}|^{m_1} \cdots |a_{l_p}|^{m_p} < \frac{M}{|A_n|}. \tag{40}$$

Let p and $l_1, \cdots l_p$ be fixed, and let us take p positive constants: $\alpha_1, \alpha_2, \cdots \alpha_p$ with $\sum_1^p \alpha_i = 1$, and, for each positive integer n, let us set $m_j = E(n\alpha_j) (1 \leqq j \leqq p-1)$, $m_p = n - \sum_1^{p-1} m_j$. Then, by Stirling's formula,

$$\lim_{n=\infty} \sqrt[n]{\frac{n!}{m_1! \cdots m_p!}} = \alpha^{-\alpha_1} \cdots \alpha_p^{-\alpha_p},$$

and by (40):

(41)

$$\alpha_1^{-\alpha_1} \cdots \alpha_p^{-\alpha_p} |a_{l_1}|^{\alpha_1} \cdots |a_{l_p}|^{\alpha_p} \leqq \frac{1}{\limsup |A_n|^{\frac{1}{n}}} = c = \min(|a|, |b|).$$

This is seen on putting in (40) $n = n_i$ where $\{n_i\}$ is such that $\lim_{i=\infty} |A_{n_i}|^{\frac{1}{n_i}} = \limsup |A_n|^{\frac{1}{n}}$, and on making i tend to infinity. Since we can suppose that the a_n are all distinct from zero we shall put:

$$\alpha_j = \frac{|a_{l_j}|}{|a_{l_1}| + \cdots + |a_{l_p}|} (1 \leqq j \leqq p).$$

Then $\alpha_j^{\alpha_j} = \left(\frac{|a_{l_j}|}{|a_{l_1}| + \cdots + |a_{l_p}|} \right)^{\alpha_j}$, and (41) gives immediately

$$|a_{l_1}| + \cdots + |a_{l_p}| \leqq \min (|a|, |b|),$$

and since the l_j are arbitrary we shall have

$$\sum_1^\infty |a_n| \leqq \min (|a|, |b|) < A,$$

which gives a contradiction.

Aronszajn proves even more general theorems, considering meromorphic functions $f(s)$, and the elementary methods

which we just used are susceptible of furnishing results [1] which contain as particular cases many theorems of H. Bohr [1] concerning almost periodic functions of a complex variable. It is not, however, our intention to explore here any farther this interesting and important field.

Obviously Theorem XIV is more precise than Theorem XIII, but we thought it useful to prove both, since, although Theorem XIV is, likewise Theorem XIII, based on Theorem XI, the passage from Theorem XI to Theorem XII or XIII is still elementary, whereas the use of the modular function is necessary for the passage from Theorem XI to Theorem XIV.

It is of course impossible to have the conclusions of either Theorems XI, XII (or XIII) and XIV without specific hypotheses on the sequences $\{\lambda_n\}$. For instance, the principal branch of $f_1(s) = \sum_1^\infty (-1)^n n e^{-ns}$ is bounded in $D(\lambda)$ with an arbitrary $\lambda\left(0 < \lambda < \frac{1}{2}\right)$, $\sigma_A^{f_1} = 0$, and yet the $|a_n|$ are not bounded. The principal branch of $f_2(s) = \sum_1^\infty (-1)^n e^{-ns}$ is also bounded in $D(\lambda)$, $\sigma_A^{f_2} = 0$, l. u. b. $|a_n| = 1$ and yet the values taken by $f_2(s)$ not only in $D(\lambda)$ but even in the half-plane $\sigma > 0$ do not penetrate the part of the circle $|u| < 1$ with $\mathcal{R}(u) \geqq \frac{1}{2}$.

IV

FUNDAMENTAL THEOREMS ON THE ESTIMATION OF COEFFICIENTS

If $f(s)$ is given by the series (1), with an abscissa of absolute convergence $\sigma_A^f < \infty$, Theorem IX allows an immediate estimation of the coefficients a_n by means of numbers by which $|f(s)|$ is bounded on a line: $\sigma = \sigma_1 > \sigma_A^f$.

If $M(t_0, \sigma_1) = \text{l. u. b.}_{t_0 \leqq t} |f(\sigma_1 + it)|$, formula (15) gives readily:

$$|a_n| e^{-\lambda_n \sigma_1} \leqq M(t_0, \sigma_1).$$

The same estimate still holds if $f(s)$ is holomorphic and bounded for $\sigma \geqq \sigma_1$, $t \geqq t_0$, since by Cauchy's theorem it follows immediately from (15) that this formula still holds with σ_1 and t_0 just defined (supposing of course that $\sigma_A^f < \infty$).

In this chapter we shall show that a very useful estimate of the coefficients can be obtained if an estimate of the maximum of $|f(s)|$ is known in a circle. But this is true only if certain conditions bearing on the distribution of the elements of $\{\lambda_n\}$ are supposed to be satisfied. The radius of such a circle has to be of a certain length which depends, again, on the distribution of the λ_n. As a matter of fact, for such an estimate of the coefficients we shall not even be obliged to suppose that the series (1) converges. It will be sufficient to suppose that the series (1) represents $f(s)$ in a certain asymptotic manner in a part of the s-plane. Considerations of this type permit us not only to prove important theorems on the distribution of singularities of a function represented by a Dirichlet series as well as theorems on the distribution of the values taken by $f(s)$ in certain

regions, but they allow us also to prove important theorems on quasi-analyticity. These considerations thus allow us to form a theory which unites in one principle both theories: that of the study of the general properties of a Dirichlet series (singularities, theorems of Picard's type, etc.) and that of quasi-analyticity. Although we give here the general theorem on the estimation of coefficients, we shall use it here only for Dirichlet series: for the other problems related to quasi-analyticity we shall refer the reader to our paper [**11**].

The number $N(x)$, $(x>0)$, of quantities λ_n smaller than x, will be called the *distribution function*, or for short, the *distribution* of $\{\lambda_n\}$. We shall suppose in this chapter that $\lambda_1>0$. Thus $N(x)=0$ for $x\leqq\lambda_1$. The quantity

$$D=\limsup_{x=\infty}\frac{N(x)}{x}$$

will be called the *upper density of* $\{\lambda_n\}$. The essential hypothesis in all the results of this chapter will be that the *upper density of* $\{\lambda_n\}$ *is finite*: $0\leqq D<\infty$. If $N(x)=Dx+n(x)$, the function $n(x)$ will be called the *excess distribution function*, or *excess distribution* of the sequence $\{\lambda_n\}$. In order to characterize the sequence more precisely than by its upper density we shall introduce an analytic function, the growth of which is closely related to the behavior of the excess distribution.

Let us set

$$\Lambda(z)=\prod_{n=1}^{\infty}\left(1-\frac{z^2}{\lambda_n^2}\right)=\sum(-1)^k c_k z^{2k}. \tag{42}$$

This product converges uniformly in each closed region and represents therefore an entire function of $z=x+iy=re^{i\varphi}$. We have $\operatorname{Max}_{|z|=r}|\Lambda(z)|=\Lambda(ir)=\prod\left(1+\frac{r^2}{\lambda_n^2}\right)$. And, by integration by parts, we get

$$\log \max_{|z|=r} |\Lambda(z)| = \log \Lambda(ir) = \sum \log\left(1+\frac{r^2}{\lambda_n^2}\right)$$

(43)

$$= \int_0^\infty \log\left(1+\frac{r^2}{x^2}\right) dN(x) = 2r^2 \int_0^\infty \frac{N(x)dx}{x(x^2+r^2)}.$$

From this equality it follows readily that

$$\limsup_{r=\infty} \frac{\log \Lambda(ir)}{r} \leqq \pi D. \tag{44}$$

We shall set

$$\Lambda^*(u) = \int_0^\infty e^{-\pi(D+u)r} \Lambda(ir)dr. \tag{45}$$

From (44) it follows that $\Lambda^*(u)$ is defined by (45) for $u>0$.

Since we have:

$$\Lambda(ir) = \sum_0^\infty c_k r^{2k},$$

where the coefficients c_k are positive, $c_0=1$, we see from (45) that

$$\Lambda^*(u) = \int_0^\infty e^{-\pi(D+u)r} \left(\sum c_k r^{2k}\right) dr = \sum_0^\infty \frac{(2k)!\, c_k}{[\pi(D+u)]^{2k+1}}. \tag{46}$$

Thus $\Lambda^*(u)$ is holomorphic and is represented by (46) for u complex with $|D+u|>D$, but we shall only use this function for $u>0$. It is positive and increases as u decreases to zero. From (43) it follows that

$$\log \Lambda(ir) = \pi D r + 2r^2 \int_0^\infty \frac{n(x)dx}{x(x^2+r^2)},$$

and (45) gives immediately:

$$\Lambda^*(u) = \int_0^\infty e^{-\pi ur + 2r^2 \int_0^\infty \frac{n(x)dx}{x(x^2+r^2)}} dr.$$

Thus, only the function $n(x)$ enters in the definition of $\Lambda^*(u)$, and this is the reason we shall call $\Lambda^*(u)$ the *growth*

function of the excess distribution $(n(x))$ *of the sequence* $\{\lambda_n\}$. Let us also set

$$\Lambda_n = \frac{1}{\lambda_n |\Lambda'(\lambda_n)|}, \quad (n \geqq 1), \tag{47}$$

(where $\Lambda'(z)$ is the derivative of $\Lambda(z)$). If we set

$$\Lambda_j(z) = \frac{\Lambda(z)}{\left(1 - \frac{z^2}{\lambda_j^2}\right)} = \sum (-1)^k C_k^{(j)} z^{2k}, \tag{48}$$

we see, since $\Lambda(\lambda_j) = 0$, that

$$\Lambda_j(\lambda_j) = \lim_{z = \lambda_j} \frac{\Lambda(z)}{\left(1 - \frac{z^2}{\lambda_j^2}\right)} = \frac{-\Lambda'(\lambda_j)\lambda_j}{2},$$

that is to say:

$$\Lambda_j = \frac{1}{2|\Lambda_j(\lambda_j)|}. \tag{49}$$

The sequence $\{\Lambda_n\}$ given by (47), or, what amounts to the same thing, by (49), shall be called the *sequence associated with the sequence* $\{\lambda_n\}$.

Let now $\{\varphi_n(x)\}$ be a sequence of non-negative continuous functions defined for $x \geqq 0$, each of these functions being non-decreasing, with $\varphi_1(0) = 0$, and satisfying, for every n, the inequality

$$\varphi_{n+1}(x) = o(\varphi_n(x)), \tag{50}$$

as x tends to 0. The function $\varphi(x) = \text{g. l. b.}_{1 \leqq n} \varphi_n(x)$ shall be called the *envelope of the sequence* $\{\varphi_n(x)\}$. Since $\varphi_n(0) = 0$ for $n \geqq 1$ (by $\varphi_1(0) = 0$ and (50)), we have $\varphi(0) = 0$. The function $\varphi(x)$ is the limit of non-negative, non-decreasing functions $\varphi_n^*(x) = \min_{1 \leqq k \leqq n} \varphi_k(x)$ tending to $\varphi(x)$ monotonically (decreasing), and therefore the integral $\int_a^b \log \varphi(x) dx$, where $0 < a < b < \infty$, has either a finite value or the value $-\infty$.

This integral can have the value $-\infty$ only if for a certain c with $a \leqq c$, $\varphi(x) = 0$, for $0 < x < c$. A sequence $\{\varphi_n(x)\}$ having the properties just described shall be called *an asymptotic sequence.*

We shall denote by $C(w, R)$ the open circle $|s-w| < R$, and by $S(v, R)$ the region $|t-t'| < R$, $\sigma > \sigma'$, where $v = \sigma' + it'$. The region $S(v, R)$ will be called a *horizontal strip of width* $2R$. If L is a Jordan arc, we shall call the union of circles $\bigcup_{s' \epsilon L} C(s', R)$ *a channel of width* $2R$. The curve L will be called the *central line of the channel.* If v and w are the extremities of L and if $R_0 \leqq R$, $R_0 \leqq R'$, we shall say *that the* corresponding *channel of width* $2R_0$ *connects the circle* $C(w, R)$ *to the horizontal half strip* $S(v, R')$.

If a function $F(s)$, holomorphic in a horizontal half-strip $S(v, R)$, and two sequences $\{a_n\}$ of complex quantities, and $\{\lambda_n\}$ with $0 < \lambda_1 < \lambda_2, \cdots \lim \lambda_n = \infty$, together with an asymptotic sequence $\{\varphi_n(x)\}$ satisfy the inequalities:

$$|F(s) - \sum_1^n a_m e^{-\lambda_m s}| \leqq \varphi_n(e^{-\sigma}),$$

where $s \epsilon S(v, R)$, with $\sigma > \sigma_1$, and where $n \geqq 1$ (σ_1 independent of n), we shall say that $\sum a_n e^{-\lambda_n s}$ *represents* $F(s)$ *in* $S(v, R)$ *asymptotically with respect to the sequence* $\{\varphi_n(x)\}$.

We shall prove the following fundamental theorem:

THEOREM XV. *Let* $F(s)$ *be a holomorphic and bounded function in a region* Δ *composed of a circle* $C(s_1, \pi d)$, *of a horizontal half-strip* $S(s_2, \pi d_2)$ *and of a channel connecting them, of width* $2\pi d_1$, $(d_1 \leqq d,\ d_1 \leqq d_2)$. *Suppose that* $F(s)$ *is represented in* $S(s_2, \pi d_2)$ *by* $\sum a_n e^{-\lambda_n s}$ *asymptotically with respect to an asymptotic sequence* $\{\varphi_n(x)\}$, *where* $\{\lambda_n\}$ *is of finite upper density* D.

If $D < d_1$, *and if, on setting* $\omega = (2(d_2 - D))^{-1}$ *and, on denoting by* $\varphi(x)$ *the envelope of* $\{\varphi_n(x)\}$ *and by* $\Lambda^*(u)$ *the growth*

function of the excess distribution of $\{\lambda_n\}$, *one of the following conditions I, II is satisfied*:

(I) *there exists a positive constant* p *such that*

$$\Lambda^*(pu^\omega)\varphi(u) = O(1)\ (u \to 0) \tag{51}$$

$$\liminf_{t=0+} \int_t^1 \log(\Lambda^*(pu^\omega)\varphi(u))u^{\omega-1}du = -\infty, \tag{52}$$

(II) *there exists a constant* $\omega' > \omega$ *such that*

$$\int_0^1 \log \varphi(u)u^{\omega'-1}du = -\infty, \tag{53}$$

then

$$|a_n| \leqq 2\pi d\Lambda^*(d-D)M(s_1)\Lambda_n e^{\lambda_n \sigma_1},\ (n \geqq 1), \tag{54}$$

where $M(s_1)$ *is the maximum of* $|F(s)|$ *in* $C(s_1, \pi d)$, *where* $\{\Lambda_n\}$ *is the sequence associated with the sequence* $\{\lambda_n\}$ *and where* σ_1 *is the real part of* s_1.

It should be remarked that (52) can hold only if

$$\int_0^1 \log \varphi(u)u^{\omega-1}du = -\infty,$$

since the function $\Lambda^*(u)$ increases as u tends to $0\ (u>0)$, and $\int_0^1 \log \Lambda^*(pu^\omega)u^{\omega-1}du > -\infty$. Therefore the condition (53) is much more restrictive than the condition (52).

For the proof of the theorem we need to prove some lemmas.

Lemma III. *If* $\Phi(s)$ *is holomorphic in a circle* $C(s', \pi c)$ *and satisfies there the inequality* $|\Phi(s)| < M$, *if* $\Lambda(z)$ *and* $\Lambda_j(z)$ *are respectively defined by* (42) *and* (48), *where the upper density of* $\{\lambda_n\}$ *satisfies the inequality* $D < c$, *then the series*

$$\sum (-1)^k c_k^{(j)} \Phi^{(2k)}(s)$$

converges uniformly in every circle $C(s', \pi(c-D-u))$, *with* $0 < u < c-D$, *and represents there a holomorphic function* $\Phi_j(s)$ *satisfying the inequality*

$$|\Phi_j(s)| < \pi c M \Lambda^*(u),$$

where $\Lambda^*(u)$ *is the growth function of the excess distribution of* $\{\lambda_n\}$.

By Cauchy's theorem we have for $s \in C(s', \pi(c-D-u))$:

$$|\Phi^{(2k)}(s)| \leqq \frac{(2k)!}{2\pi}\left|\oint_{|w-s'|=\pi c'} \frac{\Phi(w)dw}{(w-s)^{2k+1}}\right| \leqq \frac{\pi(2k)!Mc'}{[\pi(c'+D+u-c)]^{2k+1}},$$

where c' is any quantity such that $c-D-u<c'<c$. Thus

$$|\Phi^{(2k)}(s)| \leqq \frac{\pi(2k)!Mc}{[\pi(D+u)]^{2k+1}}. \tag{55}$$

On the other hand, it is clear from the definition of the quantities c_k and $c_k^{(j)}$ ((42), (48)), that $0<c_k^{(j)} \leqq c_k (k \geqq 0)$, and therefore, by (55) and (46):

$$|\Phi_j(s)| \leqq \sum_0^\infty c_k |\Phi^{(2k)}(s)| \leqq \pi c M \sum \frac{(2k)!c_k}{[\pi(D+u)]^{2k+1}} = \pi c M \Lambda^*(u).$$

We prove now a well known statement.

LEMMA IV. *If the function $F(z)$ is holomorphic and bounded in the half-plane $x \geqq a > -\infty$ and if $F(z)$ is not identically zero, then, for each $b>0$, the integral*

$$\int_{-\infty}^{\infty} \frac{\log|F(a+iy)|}{b^2+y^2}dy$$

converges to a finite value.

Let us set $F_1(z) = \dfrac{F(z+a-\frac{1}{2})}{M}$, where M is such that $|F(z)|<M$ in $x \geqq a$. $F_1(z)$ is holomorphic for

$$x \geqq \frac{1}{2},\ |F_1(z)|<1\left(x \geqq \frac{1}{2}\right),\ F_1(z) \not\equiv 0.$$

If $0<l<1$, let us denote by c_l the circle

$$\left|\frac{1-z}{z}\right| = l. \tag{56}$$

The point $z=1$ is inside of every C_l, C_l itself being inside the half-plane $x>\frac{1}{2}$. The center of C_l is the point $z_l = \dfrac{1}{1-l^2}$; its radius is $R_l = \dfrac{l}{1-l^2}$. We shall choose only such values of l that $F_1(z)$ has no zeros on the circumference C_l. There

exists obviously an infinity of such l tending increasingly to 1, since, if this were not true, the zeros of $F_1(z)$ would have a limit point in $x \geqq \frac{1}{2}$ (different from infinity), and would be identically zero, contrary to the hypotheses. If $z=1$ is a zero for $F_1(z)$ of order m_0, we shall write

$$F_1(z) = \Phi_l(z)(z-1)^{m_0}(z-\alpha_1)^{m_1}\cdots(z-\alpha_{n(l)})^{m_{n(l)}}, \tag{57}$$

where the α_i are the other zeros (of order m_i) of $F_1(z)$ in C_l. If $F_1(1) \neq 0$, $m_0 = 0$.

The function $\Phi_l(z)$ is holomorphic in C_l (closed), without zeros in this closed circle. On putting $p_l = z_l - 1 = \frac{l^2}{1-l^2}$ we have, by Poisson's integral formula, and by (56):

$$\log|\Phi_l(1)| = \frac{1}{2\pi}\int_0^{2\pi} \log|\Phi_l(z_l + R_l e^{i\varphi})| \frac{R_l^2 - \rho_l^2}{R_l^2 + 2R_l\rho_l\cos\varphi + \rho_l^2}$$

$$= \frac{l}{2\pi}\oint_{C_l} \log|\Phi_l(z)| \frac{|dz|}{|1-z|^2} = \frac{1}{2l\pi}\oint_{C_l} \log|\Phi_l(z)| \frac{|dz|}{|z|^2}. \tag{58}$$

But, if $\alpha \in C_l$, then $\left|\frac{1-\alpha}{\alpha}\right| < l$, and :

$$\oint_{C_l} \frac{\log|z-\alpha|}{|z|^2}|dz| = 2\pi(l\cdot\log l + l\log|\alpha|) \geqq 2\pi l\cdot\log|1-\alpha|. \tag{59}$$

The formulas (57), (58), and (59) give thus immediately:

$$\oint_{C_l} \log|F_1(z)| \frac{|dz|}{|z|^2} = \oint_{C_l} (\log|\Phi_l(z)| + m_0\log|z-1|$$

$$+ \sum_1^n m_i \log|z-\alpha_i|) \frac{|dz|}{|z|^2} \geqq 2\pi l \log|\Phi_l(1)| + 2\pi m_0 l \log l$$

$$+ 2\pi l \sum_1^n m_i \log|1-\alpha_i| = 2\pi l(m_0 \log l + \log|F_2(1)|),$$

where $F_2(z) = \frac{F(z)}{(z-1)^{m_0}}$. Obviously $F_2(1) \neq 0$. Since $0 < l < 1$, we see that:

$$\oint_{C_l} \log|F_1(z)| \frac{|dz|}{|z|^2} > A > -\infty,$$

where A is a constant, independent of l. Since $|F_1(z)|<1$ in $x\geqq\frac{1}{2}$,

$$\int_L \log|F_1(z)|\,\frac{|dz|}{|z|^2}>A,$$

where L is an arbitrary arc of C_l. On the other hand, the arbitrary positive quantities T, ϵ given, there exists a number l $(0<l<1)$ such that each point of an arc of C_l is at a distance smaller than ϵ from a point of the segment $\left(x=\frac{1}{2},\ |y|\leqq T\right)$, which proves that

$$\int_{-T}^{T}\log|F_1\left(\frac{1}{2}+iy\right)|\,\frac{dy}{\left(\frac{1}{2}+iy\right)^2}\geqq A. \tag{60}$$

Since (60) holds for each T (A being fixed), our lemma follows then easily.

The next lemma is a corollary of Lemma IV.

Lemma V. *If the function $\Phi(s)$ is holomorphic and bounded in the region $R(\alpha,\beta)$ defined by*

$$|t|<\frac{\pi}{2}\,(1-\alpha e^{-\sigma}),\quad \sigma\geqq\beta,$$

where $0\leqq\pi\alpha<2e^{\beta}$, and is not identically zero, then

$$\sigma+i\cos^{-1}e^{\beta-\sigma}\ \epsilon\ R(\alpha,\beta)$$

for $\sigma>\beta$ (here $0\leqq\cos^{-1}e^{\beta-\sigma}<\frac{\pi}{2}$) and

$$\int_{\beta}^{\infty}\log|\Phi(\sigma+i\cos^{-1}e^{\beta-\sigma})|\,e^{-\sigma}d\sigma>-\infty. \tag{61}$$

If s belongs to the boundary of $R(\alpha,\beta)$, and if $\sigma>\beta$, then

$$\cos t=\sin\left(\frac{\pi}{2}-|t|\right)\leqq\frac{\pi}{2}-|t|=\frac{\pi\alpha}{2e^{\sigma}}<e^{\beta-\sigma},$$

and therefore, if $\cos t=e^{\beta-\sigma}$, $\sigma\geqq\beta$, then $s=\sigma+it\ \epsilon\ R(\alpha,\beta)$, which is equivalent to the assertion that

$$s=\sigma+i\cos^{-1}(e^{\beta-\sigma})\ \epsilon\ R(\alpha,\beta)$$

for $\sigma \geqq \beta$. The curve C, in the s-plane, on which $\cos t = e^{\beta-\sigma}$ $(\sigma \geqq \beta)$, and the region bounded by this curve, in which $\cos t > e^{\beta-\sigma}$ $\left(|t| < \frac{\pi}{2}, \sigma > \beta\right)$, are mapped by the transformation $z = x+iy = e^s$ into the straight-line $x = e^\beta = \gamma$ and the half-plane $x > \gamma$. The function of z, $f(z) = \Phi(s)$, is then, by hypothesis, holomorphic and bounded for $x \geqq \gamma$, and is not identically zero.

It follows then from Lemma IV that

$$\int_0^\infty \frac{\log|f(\gamma+iy)|}{1+y^2}\,dy > -\infty,$$

and from $y = e^\sigma \sin t$, $dy = (e^\sigma \sin t + e^\beta \operatorname{cotn} t)\,d\sigma$, as $s \in C$, it follows that

(62)

$$\int_{\beta+1}^\infty \log|\Phi(\sigma + i\cos^{-1}e^{\beta-\sigma})|\,(e^\sigma \sin t + e^\beta \operatorname{cotn} t)\,\frac{d\sigma}{1+e^{2\sigma}\sin^2 t} > -\infty.$$

From the obvious relationship

$$(e^\sigma \sin t + e^\beta \operatorname{cotn} t)(1+e^{2\sigma}\sin^2 t)^{-1} \sim e^{-\sigma},$$

as s, on C, tends to infinity $\left(\sigma \text{ tends then to } +\infty \text{ and } t \text{ to } \frac{\pi}{2}\right)$, and from (62) follows then (61).

We now proceed to the proof of Theorem XV.

Let L be the central line of the channel connecting $C(s_1, \pi d)$ to $S(s_2, \pi d_2)$, $(s_2 = \sigma_2 + it_2)$, and belonging to Δ. Let $F(s) = \sum(-1)^k c_k^{(j)} F^{(2k)}(s)$, where the $c_k^{(j)}$ are given by (48). By the hypotheses of the theorem and by Lemma III, $F_j(s)$ is holomorphic in each of the circles $C(s_1, \pi(d-D-u))$, $C(s', \pi(d_1-D-u))(s' \in L)$, $C(\sigma+it_2, \pi(d_1-D-u))(\sigma_2 < \sigma \leqq \sigma_2 + \pi d_2)$, $C(\sigma+it_2, \pi(d_2-D-u))(\sigma > \sigma_2+\pi d_2)$, where u is such that $0 < u < d_1 - D$.

We have by the same Lemma III:

$$|F_j(s_1)| < \pi d M(s_1)\Delta^*(u),$$

where u is such that $0<u<d-D$. Therefore, since $\Lambda^*(u)$ is continuous for $u>0$:

$$|F_j(s_1)| \leqq \pi d M(s_1)\Lambda^*(d-D). \tag{63}$$

Let us now write

$$\Phi_n(s) = F(s) - \sum_1^n a_m e^{-\lambda_m s}. \tag{64}$$

By hypotheses, there exists a number σ' such that $\Phi_n(s)(n \geqq 1)$ is holomorphic in every circle $C(\bar{\sigma}+it_2,\ \pi d_2)$ with $\bar{\sigma} \geqq \sigma'$, and in every such circle:

$$|\Phi_n(s)| \leqq \varphi_n(e^{-\bar{\sigma}+\pi d_2}),\ (n \geqq 1), \tag{65}$$

(σ' is independent of n).

It follows then from Lemma III that the function

$$\Phi_{n,j}(s) = \sum_{n=0}^{\infty}(-1)^k c_k^{(j)}\Phi_n^{(2k)}(s) = \sum_k (-1)^k c_k^{(j)} F^{(2k)}(s) -$$

$$\sum_k (-1)^k c_k^{(j)} \sum_{m=1}^n \lambda_m^{2k} a_m e^{-\lambda_m s} = F_j(s) - \tag{66}$$

$$\sum_{m=1}^n a_m e^{-\lambda_m s} \sum_n (-1)^k c_k^{(j)} \lambda_m^{2k} = F_j(s) - \sum_{m=1}^n a_m \Lambda_j(\lambda_m) e^{-\lambda_m s}$$

is holomorphic in every circle $C(\bar{\sigma}+it_2,\ \pi(d_2-D-u))$, $\bar{\sigma} \geqq \sigma'$ with $0<u<d_2-D$, and in this circle:

$$|\Phi_{n,j}(s)| \leqq \pi d_2 \varphi_n(e^{-\bar{\sigma}+\pi d_2})\Lambda^*(u). \tag{67}$$

Since $\Lambda_j(\lambda_m)=0$ for $m \neq j$, we see that for $n \geqq j$:

$$\sum_{m=1}^n a_m \Lambda_j(\lambda_m) e^{-\lambda_m s} = a_j \Lambda_j(\lambda_j) e^{-\lambda_j s},$$

and we may write, for $n \geqq j$ (by (66) and (67)):

$$\Phi_{n,j}(s) = \Psi_j(s) = F_j(s) - a_j \Lambda_j(\lambda_j) e^{-\lambda_j s}, \tag{68}$$

$$|\Psi_j(s)| \leqq \pi d_2 \varphi_n(e^{-\sigma+\pi d_2})\Lambda^*(u) \tag{69}$$

$(s=\sigma+it,\ |t-t_2| \leqq \pi(d_2-D-u),\ 0<u<d_2-D,\ \sigma>\sigma',\ j \leqq n)$.

From $\varphi_{n+1}(x) = o(\varphi_n(x))(x \to 0)$ it follows that for σ sufficiently large and j fixed $\varphi_k(e^{-\sigma+\pi d_2}) \geqq \varphi_j(e^{-\sigma+\pi d_2})$, $1 \leqq k \leqq j$. Therefore there exists a quantity σ^* independent of n such that (69) holds for $|t-t_2| \leqq \pi(d_2-D-u)$, $\sigma \geqq \sigma^*$, $0<u<d_2-D$, $n \geqq 1$. We have hence, for the same values of s,

$$|\Psi_j(s)| \leqq \pi d_2 \Lambda^*(u) \text{g. l. b.}_{n \geqq 1} \varphi_n(e^{-\sigma+\pi d_2}) = \pi d_2 \Lambda^*(u) \varphi(e^{-\sigma+\pi d_2}). \tag{70}$$

We shall now prove that $\Psi_j(s)$ is identically zero. For that purpose we shall consider the function

$$\theta_j(\nu; s) = \Psi_j\left(\frac{s}{\nu} + it_2\right),$$

where $\nu \geqq \omega$. It follows from (70) that, in the region

$$|t| \leqq \pi\nu(d_2 - D - u) = \frac{\pi}{2}\left(\frac{\nu}{\omega} - 2\nu u\right), \quad \sigma \geqq \nu\sigma^*$$

with $0 < u < d_2 - D$, the following inequality holds:

$$|\theta_j(\nu; s)| \leqq \pi d_2 \Lambda^*(u) \varphi(e^{-\frac{\sigma}{\nu}+\pi d_2}). \tag{71}$$

Let q be an arbitrary positive quantity and let us set $\alpha = \alpha(q) = 2q\omega e^{\pi d_2 \omega}$, $\beta = \beta(q) > \max\left(\nu\sigma^*, \log\frac{\pi\alpha}{2}\right)$. We have then $\beta > \nu\sigma^*$, $2e^{\beta} > \pi\alpha > 0$. Since in (71) u can be taken arbitrarily from the interval $(0, d_2 - D)$, we set for $\sigma \geqq \beta$: $u = \frac{1}{2\omega} - \frac{1}{2\nu} + \frac{\alpha}{2\nu e^{\sigma}}$. The condition $0 < u < d_2 - D = \frac{1}{2\omega}$ is then satisfied, since $\nu > \omega$ and $1 - \alpha e^{-\beta} > 0$. It follows from (71) that $\theta_j(\nu; s)$ is holomorphic in the region $R(\alpha, \beta)$ given by $|t| < \frac{\pi}{2}(1 - \alpha e^{-\sigma})$, $\sigma \geqq \beta$, and satisfies there the inequality:

$$|\theta_j(\nu; s)| \leqq \pi d_2 \Lambda^*\left(\frac{1}{2\omega} - \frac{1}{2\nu} + \frac{\omega}{\nu} q e^{\pi d_2 \omega - \sigma}\right) \varphi\left(e^{-\frac{\sigma}{\nu}+\pi d_2}\right). \tag{72}$$

This inequality shows first that, if $\nu > \omega$, $\theta_j(\nu; s)$ is bounded in $R(\alpha, \beta)$. Indeed, as $\sigma \to \infty$, the right member of (72) tends to $\pi d_2 \Lambda^*\left(\frac{1}{2\omega} - \frac{1}{2\nu}\right)\varphi(0) = 0$. But, if, moreover, (51) of the condition (I) of the theorem is satisfied, then $\theta_j(\omega; s)$ is also bounded in $R(\alpha, \beta)$ provided that in the definition of α and β, given above, q takes the value p involved in (51).

Indeed, by (72), we have, in the region $R(\alpha, \beta)$ so defined $(\alpha=\alpha(p),\ \beta=\beta(p))$:

$$(73)\quad |\theta_j(\omega; s)| \leqq \pi d_2 \Lambda^*\left(pe^{-\omega\left(\frac{\sigma}{\omega}-\pi d_2\right)}\right)\varphi\left(e^{-\left(\frac{\sigma}{\omega}-\pi d_2\right)}\right),$$

and by (51) we see that $|\theta_j(\omega; s)| < N < \infty$ in $R(\alpha, \beta)$. Thus we have proved that $\theta_j(\nu; s)$ is bounded in $R(\alpha(q), \beta(q))$ with $q>0$, arbitrary, and that $\theta(\omega; s)$ is bounded in $R(\alpha(p), \beta(p))$. If $\Psi_j(s)$ were not identically zero, $\theta_j(\nu; s)$ would be not identically zero for ν arbitrary, and, by Lemma V we would have:

$$(74)\qquad \int_\beta^\infty \log|\theta_j(\nu; \sigma + i\cos^{-1}e^{\beta-\sigma})|\, e^{-\sigma}d\sigma > -\infty,$$

with $\beta=\beta(1)$, if $\nu>\omega$, and with $\beta=\beta(p)$, if $\nu=\omega$. From (74) and (72), both with $\nu>\omega$, $q=1$, it follows that for $k>0$ sufficiently large:

$$(75)\quad \begin{aligned}&\log\Lambda^*\left(\frac{1}{2\omega}-\frac{1}{2\nu}\right)e^{-k}+\int_k^\infty \log\varphi(e^{-\frac{\sigma}{\nu}+\pi d_2})e^{-\sigma}d\sigma\\ &\geqq \int_k^\infty \log\left[\Lambda^*\left(\frac{1}{2\omega}-\frac{1}{2\nu}+\frac{\omega}{\nu}e^{\pi d_2\omega-\sigma}\right)\varphi(e^{-\frac{\sigma}{\nu}+\pi d_2})\right]e^{-\sigma}d\sigma > -\infty.\end{aligned}$$

If condition (II) is satisfied, and if we choose $\nu=\omega'$, we shall have by (75), on putting $e^{-\frac{\sigma}{\omega'}+\pi d_2}=u$:

$$\int_0^K \log\varphi(u)u^{\omega'-1}du > -\infty \quad (K \text{ constant} > 0),$$

which is in contradiction with (53). Thus if condition (II) is satisfied, $\Psi_j(s)\equiv 0$.

From (74) and (73) it follows, on putting $e^{-\frac{\sigma}{\omega}+\pi d_2}=u$, that

$$\liminf_{t=0+}\int_t^1 \log[\Lambda^*(pu^\omega)\varphi(u)]u^{\omega-1}du > -\infty,$$

which is in contradiction with (52). Thus if condition (I) is satisfied we still have $\Psi_j(s)\equiv 0$. In other words: if the conditions of the theorem are satisfied we have

$$(76)\qquad F_j(s)\equiv a_j\Lambda_j(\lambda_j)e^{-\lambda_j s}.$$

It follows then from (63) that

(77) $$|a_j\Lambda_j(\lambda_j)|e^{-\lambda_j\sigma_1} \leqq \pi d M(s_1)\Lambda^*(d-D),$$

which is equivalent to (54). Theorem XV is then completely proved.

The following lemma will allow us to consider Dirichlet series with an abscissa of absolute convergence ($\sigma_A < \infty$) as representing the principal branch of the function represented by this series, asymptotically with respect to a simple asymptotic sequence $\varphi_n(x)$, for which both conditions (I) and (II) of Theorem XV are automatically satisfied.

LEMMA VI. *If $\sum a_n e^{-\lambda_n s}$ has an abscissa of absolute convergence $\sigma_A < \infty$, then the principal branch of the function $F(s)$ represented by this series is represented asymptotically by this series with respect to the asymptotic sequence*

$$\varphi_n(x) = A(\sigma')e^{\lambda_n\sigma'}x^{\lambda_n}, \quad A(\sigma') = \sum |a_n| e^{-\lambda_n\sigma'}, \quad \sigma' > \sigma_A,$$

in each horizontal half-strip $S(v, R)$ with $v = \sigma_1 + it_1$, $\sigma_1 \geqq \sigma' > \sigma_A$, t_1 and R arbitrary.

We have indeed for $\sigma > \sigma'$:

$$|F(s) - \sum_{m=1}^{n} a_m e^{-\lambda_m s}| \leqq \sum_{n+1}^{\infty} |a_m| e^{-\lambda_m\sigma} = e^{-\lambda_n\sigma} \sum_{n+1}^{\infty} |a_m| e^{-(\lambda_m-\lambda_n)\sigma}$$

$$\leqq e^{-\lambda_n\sigma} \sum_{n+1}^{\infty} |a_m| e^{-(\lambda_m-\lambda_n)\sigma'} \leqq \Big[\sum_{m=1}^{\infty} |a_m| e^{-(\lambda_m-\lambda_n)\sigma'}\Big] e^{-\lambda_n\sigma}$$

$$= A(\sigma')e^{\lambda_n\sigma'}e^{-\lambda_n\sigma}.$$

Theorem XV together with Lemma VI gives immediately the following theorem.

THEOREM XVI. *Let $F(s)$ be holomorphic in a region composed of a circle $C(s_1, \pi d)$, of the half-plane $\sigma > \sigma_0$ and of a channel of width $2\pi d_1$, the central line of which has one extremity in the half-plane $\sigma > \sigma_0$, the other extremity being the point s_1. Let $F(s)$ be given for $\sigma > \sigma_0$ by*

$$F(s) = \sum a_n e^{-\lambda_n s},$$

the upper density of $\{\lambda_n\}$ being D. If $D < d_1 \leqq d$ then

(78) $$|a_n| \leqq 2\pi d\Lambda^*(d-D)M(s_1)\Lambda_n e^{\lambda_n\sigma_1},$$

where Λ^*, $M(s_1)$, Λ_n, σ_1 *have the same meaning as in Theorem XV.*

Since $D<\infty$, we see, by Theorem VII, that $\sigma_A{}^F \leqq \sigma_0 < \infty$, and by Lemma VI, $F(s)$ is represented asymptotically in $S(s_2, \pi d_2)$, with $s_2=\sigma_2+it_2$, $\sigma_2 \geqq \sigma' > \sigma_A{}^F$, $d_2=d$, by $\sum a_n e^{-\lambda_n s}$ with respect to $\{\varphi_n(x)\}$, where $\varphi_n(x)=A(\sigma')e^{\lambda_n\sigma'}x^{\lambda_n}$. But obviously $\varphi(x)=\underset{1\leqq n}{\text{g. l. b.}}\ \varphi_n(x)=0$ for $x<e^{-\sigma'}$, and therefore both conditions (I) and (II), each of which is sufficient for the conclusion of Theorem XV, are satisfied. The conclusion of Theorem XVI, which is that of Theorem XV, is therefore satisfied.

We have seen that Theorem XVI follows immediately from Theorem XV, but Theorem XVI can be proved much more simply than Theorem XV itself.

In order to prove Theorem XVI directly it is sufficient, after having proved Lemmas III and VI (Lemmas IV and V are here unnecessary), to proceed with the proof exactly as for Theorem XV, with $\varphi_n(x)=A(\sigma')e^{\lambda_n\sigma'}x^{\lambda_n}$, until the inequality (69) is established. From this inequality it follows immediately that:

(69′) $$|\Psi_j(s)| \leqq \pi d_2\Lambda^*(u)\ \underset{j\leqq n}{\text{g. l. b.}}\ \varphi_n(e^{-\sigma+\pi d_2})=0,$$

for $\sigma>\pi d_2+\sigma'$. This gives then (76) and (77). In other words, all the reasoning which is extended from the inequality (69) to the equality (76) is replaced by the establishment of (69′) as an immediate consequence of (69).

The theorems of this chapter were established by the author [**11**].

V

ESTIMATION OF THE SEQUENCE ASSOCIATED WITH THE SEQUENCE OF EXPONENTS

Both Theorems XV and XVI can be used with more precision if an estimate of the growth of the sequence $\{\Lambda_n\}$ can be given in relation to the growth of the sequence $\{\lambda_n\}$.

We shall thus prove the following two theorems.

THEOREM XVII. *If the upper density of the sequence* $\{\lambda_n\}$ *is* D, *and if*

$$\liminf_{n=\infty} (\lambda_{n+1}-\lambda_n) = h > 0, \tag{79}$$

then to every $\epsilon > 0$ *there corresponds a quantity* $A(\epsilon)$ *such that, on denoting by* $\{\Lambda_n\}$ *the sequence associated with* $\{\lambda_n\}$, *we have*

$$\Lambda_n \leqq A(\epsilon) e^{(B(D,h)+\epsilon)\lambda_n}, \quad (n \geqq 1), \tag{80}$$

where

$$B(D, h) = 3D(3 - \log(hD)) \quad (\text{with } B(0, h) = \lim_{D=0} B(D, h) = 0).$$

REMARK. It is obvious from the definition of D and h that $Dh \leqq 1$.

THEOREM XVIII. *If the upper density of the sequence* $\{\lambda_n\}$ *is finite, and if there exists a finite positive quantity* μ *such that for* n *sufficiently large:*

$$\lambda_{n+1} - \lambda_n > \lambda_n^{-\mu}, \tag{81}$$

then to every $\epsilon > 0$ *there corresponds a quantity* $A(\epsilon)$ *such that*

$$\Lambda_n \leqq A(\epsilon) e^{\lambda_n^{1+\epsilon}}, \quad (n \geqq 1). \tag{82}$$

Theorem XVII is an immediate consequence of a theorem of A. Ostrowski [**16**], (here Lemma VII) for which we give the proof given by V. Bernstein [**3**]. Theorem XVIII is a simple consequence of a classical theorem of Borel [**19**] on canonical products (here Lemma VIII).

LEMMA VII. *If the relationship* (79) *holds, the entire function* $\Lambda(z)$ *given by* (42) *has the following property: to* $\epsilon'>0$ *and* $q>0$ *arbitrary, there corresponds a quantity* $R(\epsilon', q)$ *such that, on denoting by* D *the upper density of* $\{\lambda_n\}$,

$$|\Lambda(z)|>e^{-(B(D,h)+\epsilon')|z|},$$

for each z *satisfying the set of inequalities* $|z|>R(\epsilon', q)$, $|z\mp\lambda_n|>q$, $(n\geqq 1)$.

From (79) and from the relationship $Dh\leqq 1$ it follows that $D\leqq\frac{1}{h}<\infty$.

Let ω be an arbitrary quantity such that $0<\omega<h$. For $n>n_1=n_1(\omega)$ we have $\lambda_{n+1}-\lambda_n>h-\omega$. For $x>0$, let us set $\xi=\frac{x}{h-\omega}$. Let us also set $\mu_n=\frac{\lambda_n}{h-\omega}$, and suppose that $0<q<\frac{h-\omega}{2}$. If $|z\pm\lambda_n|\geqq q$, and if $\lambda_{m-1}\leqq x\leqq\lambda_m (z=x+iy)$, then, for $x>n_1+1$ with $n_1+2<m$:

$$\frac{1}{|\Lambda(z)|}=\prod_1^\infty\left|\frac{\lambda_n^2}{\lambda_n^2-z^2}\right|$$

$$(83)\qquad \leqq\frac{2|z|^2}{q^2}\left|\prod_1^{n_1}\left(\frac{\lambda_n^2}{\lambda_n^2-z^2}\right)\right|\cdot\prod_{n_1+1}^{m-2}\frac{\lambda_n^2}{x^2-\lambda_n^2}\prod_{m+1}^{\infty}\frac{\lambda_n^2}{\lambda_n^2-x^2}$$

$$\leqq A(\omega)|z|^2\prod_{n_1+1}^{m-2}\frac{\mu_n^2}{\xi^2-\mu_n^2}\prod_{m+1}^{\infty}\left(1+\frac{\xi^2}{(\mu_n-\xi)(\mu_n+\xi)}\right),$$

since the product extended from 1 to n_1 is bounded when $x>\lambda_{m-1}$, with $m>n_1+2$, the quantity which bounds it depending on n_1, that is to say, on ω. If $n>n_1$ we have $\mu_{n+1}-\mu_n>1$; let us set $\Delta_n=\underset{m\geqq n}{\text{l. u. b.}}\frac{m-n_1}{\mu_m}$. It is readily seen that Δ_n tends decreasingly to $\limsup_{m=\infty}\frac{m-n_1}{\mu_m}=\Delta=(h-\omega)D$. We have therefore:

$$\prod_{n_1+1}^{m-2}\frac{\mu_n^2}{\xi^2-\mu_n^2}\leqq\prod_{n_1+1}^{m-2}\frac{\mu_n}{\xi-\mu_n}\leqq\frac{\mu_{m-2}}{(m-n_1-2)!}\leqq\left(\frac{\mu_{m-2}e}{m-n_1-2}\right)^{m-n_1-2}$$

$$(84) \qquad = \left(\frac{\mu_{m-2}e}{m-n_1-2}\right)^{\frac{m-n_1-2}{\mu_{m-2}}} \leqq \left(\frac{\mu_{m-2}e}{m-n_1-2}\right)^{\frac{m-n_1-2}{\mu_{m-2}}} \xi .$$

We have $\frac{\mu_{m-2}}{m-n_1-2} \geqq \frac{1}{\Delta_{m-2}}$. Since the maximum of $x^{\frac{1}{x}}$ $(x>0)$ is at $x=e$, we see that, if $\Delta < \frac{1}{e}$, then, for m sufficiently large (in order that $\Delta_{m-2} < \frac{1}{e}$), $\left(\frac{\mu_{m-2}}{m-n_1-2}\right)^{\frac{m-n_1-2}{\mu_{m-2}}} \leqq \left(\frac{1}{\Delta_{m-2}}\right)^{\Delta_{m-2}}$, and, if $\Delta_{m-2} \geqq \Delta \geqq \frac{1}{e}$, then $\left(\frac{\mu_{m-2}}{m-n_1-2}\right)^{\frac{m-n_1-2}{\mu_{m-2}}} \leqq e^{\frac{1}{e}}$.

In the first case, $\left(\Delta_{m-2} < \frac{1}{e}\right)$, we get from (84):

$$\prod_{n_1+1}^{m-2} \frac{\mu_n^2}{\xi^2-\mu_n^2} \leqq \left(\frac{e}{\Delta_{m-2}}\right)^{\Delta_{m-2}\xi} = e^{\Delta_{m-2}\log\left(\frac{e}{\Delta_{m-2}}\right)\xi} = e^{\Delta_{m-2}(1-\log\Delta_{m-2})\xi}.$$

In the second case, $\left(\Delta_{m-2} \geqq \frac{1}{e}\right)$:

$$\prod_{n_1+1}^{m-2} \frac{\mu_n^2}{\xi^2-\mu_n^2} \leqq e^{\left(\frac{1}{e}+\Delta_{m-2}\right)\xi}.$$

Hence, since $\Delta = (h-\omega)D < 1$, and $\Delta_{m-2} < 1$ for m sufficiently large, we may write (for m large):

$$e^{\left(\frac{1}{e}+\Delta_{m-2}\right)\xi} \leqq e^{\Delta_{m-2}(2-\log\Delta_{m-2})\xi}.$$

Therefore, in each case, for m sufficiently large:

$$(85) \qquad \prod_{n_1+1}^{m-2} \frac{\mu_n^2}{\xi^2-\mu_n^2} \leqq e^{\Delta_{m-2}(2-\log\Delta_{m-2})\xi}.$$

Let us now estimate the product in (83) extended from $m+1$ to ∞. Let μ_k be the largest μ_n smaller than 2ξ, if $2\xi > \mu_{m+1}$, and let $k=m$ if $2\xi \leqq \mu_{m+1}$. Let us set $E(\xi)+1=\zeta$, $E(2\Delta_k\xi)=\alpha\zeta$. We have $\alpha\zeta \geqq 2\Delta_k\xi \geqq 2\xi\frac{k-n_1}{\mu_k} \geqq k-n_1$.

We have[1]:

[1] If $k=m$ the product $\prod_{m+1}^{k}$ should be replaced by unity.

$$Q(\xi)=\prod_{m+1}^{\infty}\left(1+\frac{\xi^2}{(\mu_n-\xi)(\mu_n+\xi)}\right)\leqq\prod_{m+1}^{\infty}\left(1+\frac{4}{3}\xi^2\left(\frac{\Delta_k}{n-k}\right)^2\right)$$

$$=\prod_{\nu=1}^{\alpha\zeta}\left(1+\frac{\xi}{\nu}\right)\prod_{\nu=1}^{\infty}\left(1+\frac{4}{3}\xi^2\left(\frac{\Delta_k}{\nu}\right)^2\right)\leqq\frac{((1+\alpha)\xi)!}{(\alpha\zeta)!\,\xi!}\,\frac{\sin(\frac{2}{\sqrt{3}}\pi i\Delta_k\xi)}{\frac{2}{\sqrt{3}}\pi i\Delta_k\xi}$$

$$\leqq B\frac{((1+\alpha)\xi)^{(1+\alpha)\zeta}}{(\alpha\zeta)^{\alpha\zeta}\zeta^{\zeta}}\,e^{\frac{2}{\sqrt{3}}\pi\Delta_k\xi}=B\left(\frac{1+\alpha}{\alpha}\right)^{\alpha\zeta}(1+\alpha)^{\zeta}\,e^{\frac{2\pi}{\sqrt{3}}\Delta_k\xi}$$

(B const.).

By definition, $\alpha<2\Delta_k$. If therefore we suppose $\Delta_k<\frac{1}{2}$, we shall have:

$$Q(\xi)\leqq Ce^{(2\alpha-\alpha\log\alpha+\frac{2}{\sqrt{3}}\pi\Delta_k)\xi}\quad(\text{C const.}).$$

The expression $2\alpha-\alpha\log\alpha$ increases as α increases from 0 to e, thus, if $\alpha<2\Delta_k\leqq 1$:

$$Q(\xi)\leqq Ce^{(4\Delta_k-2\Delta_k\log(2\Delta_k)+\frac{2\pi}{\sqrt{3}}\Delta_k)\xi}\leqq Ce^{(7\Delta_k-2\Delta_k\log\Delta_k)\xi}. \tag{86}$$

We have, on the other hand, for ξ sufficiently large:

$$Q(\xi)\leqq\prod_{n=m+1}^{\infty}\left(1+\frac{\xi^2}{\mu_n(n-m)}\right)\leqq\prod_{\nu=1}^{\infty}\left(1+\left(\frac{\xi\sqrt{\Delta_m}}{\nu}\right)^2\right)$$

$$=\frac{\sin(\pi i\xi\sqrt{\Delta_m})}{\Delta i\xi\sqrt{\Delta_m}}\leqq e^{\xi\sqrt{\Delta+2\omega}}.$$

Thus, if ω is sufficiently small, that is to say if n_1, and thus if m, is large, that is to say if ξ is sufficiently large, $Q(\xi)$ satisfies (86) also if $2\Delta_k>1$. In all cases we have therefore, by (83), (85), and (86), for x sufficiently large ($x>x_\omega$), with ω fixed arbitrarily small ($|z-\lambda_n|\geqq q$):

$$\frac{1}{|\Delta(z)|}\leqq e^{(\Delta(2-\log\Delta)+\Delta(7-2\log\Delta)+\omega)\xi}$$

$$=e^{(9\Delta-3\Delta\log\Delta+\omega)\frac{x}{h-\omega}}=e^{3D(3-\log[D(h-\omega)])x+\frac{\omega}{h-\omega}x}$$

$$\leqq e^{3D(3-\log[D(h-\omega)])|z|+\frac{\omega}{h-\omega}|z|}.$$

This proves that the desired inequality holds (since $\Delta(z)$ is even) if z is in the described region with the supplementary

condition $|x| \geqq a(\epsilon')$. But if $|x| \leqq c$, $\lambda_m < c \leqq \lambda_{m+1}$, with z satisfying the conditions of the lemma, then for $n \geqq m+2$:

$$\frac{\lambda_n^2}{|\lambda_n^2 - z^2|} \leqq \frac{\lambda_n^2}{\lambda_n^2 - x^2} \leqq \frac{\lambda_n^2}{\lambda_n^2 - c^2}.$$

This proves that in the region described in the lemma, with the supplementary condition $|x| \leqq c$:

$$\frac{1}{|\Lambda(z)|} \leqq K \left| \prod_{n=1}^{m+1} \frac{\lambda_n}{\lambda_n^2 - z^2} \right|,$$

where $K = K(c)$ is a constant. This achieves obviously the proof of Lemma VII.

The proof of Theorem XVII is now immediate. On choosing $q < h$, we see, by Lemma VII, that for n sufficiently large, in the closed circle $C(\lambda_n, q)$ (since this circle contains no λ_k with $k \neq n$) the following inequality holds

$$\left| \frac{z - \lambda_n}{\Lambda(z)} \right| < q e^{[B(D,h)+\epsilon'](\lambda_n + q)},$$

and $$\lambda_n \Lambda_n = \frac{1}{|\Lambda'(\lambda_n)|} = \lim_{z=\lambda_n} \left| \frac{z - \lambda_n}{\Lambda(z)} \right| \leqq q e^{[B(D,h)+\epsilon'](\lambda_n + q)},$$

which proves Theorem XVII.

LEMMA VIII. *Let $P(z)$ be a canonical product with zeros at the points $\{p_n\}$, the exponent of convergence of $\{|p_n|\}$ being ρ_1.*[1] *To each couple of two constants $h > \rho_1$, $\epsilon' > 0$, there corresponds a quantity $R(h, \epsilon')$ such that, if $|z - p_n| > p_n^{-h}$, $(n \geqq 1)$, $|z| > R(h, \epsilon')$, the following inequality holds:*

$$|P(z)| > e^{-r^{\rho_1 + \epsilon'}}.$$

This is a classical theorem of Borel, for the proof of which we refer the reader to Valiron's book on entire functions [**19**]. One of the reasons we do not give its proof here is the fact that it serves here only for the proof of Theorem XVIII which, though interesting in itself, will not be used during these lectures.

[1] That is to say, if $\epsilon > 0$: $\Sigma \frac{1}{|p_n|^{\rho_1 + \epsilon}} < \infty$, $\Sigma \frac{1}{|p_n|^{\rho_1 - \epsilon}} = \infty$.

Let us now proceed with the proof of Theorem XVIII. On setting $p_n = \lambda_n^2$, we see, since the upper density of $\{\lambda_n\}$ is finite, that there exists a constant $\lambda > 0$, such that $p_n > \lambda n^2$, and the exponent of convergence of $\{p_n\}$ is thus not larger than $\frac{1}{2}$. By Lemma VIII, in the region outside the circles $C(p_n, p_n^{-h})$, with $h > \frac{1}{2}$, we have

$$\left| \prod_{n=1}^{\infty} \left(1 - \frac{\zeta}{p_n}\right) \right| > e^{-|\zeta|^{\frac{1}{2}+\epsilon'}}, \tag{87}$$

if $|\zeta| > r_{\epsilon'}$.

Let us set $\nu = \max(2, \mu)$. Since for $n > n_0$, $\lambda_{n+1} - \lambda_n > \lambda_n^{-\nu}$, we have for $n > n_0 + 1$: $p_{n+1} - p_n > (\lambda_{n+1} + \lambda_n)\lambda_n^{-\nu} > 2p_n^{\frac{1-\nu}{2}}$ and $p_n - p_{n-1} > 2p_{n-1}^{\frac{1-\nu}{2}} > 2p_n^{\frac{1-\nu}{2}}$. In other words, the closed circles $C(p_n, p_n^{\frac{1-\nu}{2}})$ contain no p_k with $k \neq n$. By (87) we shall then have on the circle $C(p_n, p_n^{\frac{1-\nu}{2}})$, for n large:

$$\left| \prod_{k=1}^{\infty} \left(1 - \frac{\zeta}{p_k}\right) \right| > e^{-\left(p_n + p_n^{\frac{1-\nu}{2}}\right)^{\frac{1}{2}+\epsilon'}} > e^{-(2p_n)^{\frac{1}{2}+\epsilon'}} > e^{-\lambda_n^{1+3\epsilon'}}.$$

On the circumference of this circle we have then

$$\left| \frac{\zeta - p_n}{\prod_{k=1}^{\infty} \left(1 - \frac{\zeta}{p_k}\right)} \right| \leqq p_n^{\frac{1-\nu}{2}} e^{\lambda_n^{1+3\epsilon'}}.$$

We have thus for $n > n(\epsilon')$:

$$\lim_{\zeta = p_n} \left| \frac{\zeta - p_n}{\prod \left(1 - \frac{\zeta}{p_k}\right)} \right| = \lim_{z = \lambda_n} \left| \frac{z^2 - \lambda_n^2}{\Lambda(z)} \right| = 2\lambda_n^2 \Lambda_n \leqq \lambda_n^{1-\nu} e^{\lambda_n^{1+3\epsilon'}},$$

which proves Theorem XVIII.

In the inequalities (54) or (78) the quantities Λ_n can be replaced by the right-hand expression of the inequality (80), if the λ_n satisfy the relationship (79) (in addition, of course, to the relationships required by Theorems XV or XVI), and by the right-hand expression of the inequality (82), if the λ_n satisfy the supplementary condition (81).

VI

THEOREMS OF LIOUVILLE AND PICARD TYPE IN A CURVILINEAR STRIP

The considerations of the preceding chapter allow an extensive study of the behavior of a function represented by a Dirichlet series, or even, more generally, of a function represented asymptotically with respect to an asymptotic sequence in a strip, or in a channel of which the width depends on the upper density of the exponents. Theorems XV and XVI will play in our study the same rôle as is played in the study of general analytic functions by the well known estimate of Cauchy:

$$|a_n| < \frac{M}{R^n},$$

where $f(z) = \sum a_n z^n$, $|f(z)| \leqq M$, (where R is smaller than the radius of convergence of the Taylor series). The results obtained by these theorems are, in nature, much more general than the classical theorems, even if only Taylor series are involved, especially when "gap series" are studied. One of the simplest results is that which generalizes the famous theorem of Liouville.

Let us first give the following definition.

Suppose that $s(u) = \sigma(u) + it(u)$ is a continuous complex function of the real variable $u(-\infty < u < \infty)$. Let us suppose that for $u > u_0$: $t(u) = t_0$ (constant), and that $\sigma(u) \to -\infty$ as $u \to -\infty$; let also R be a positive quantity. The region

$$\sum = \sum (s(u), R) = \bigcup_{-\infty < u < \infty} C(s(u), R)$$

(the union of the open circles of radius R and center $s(u)$

as u takes all real values) shall be called a *curvilinear strip of width $2R$, horizontal at the right and extending to $-\infty$ at the left.* The curve composed of the points of which the affixes are given by $s(u)$ shall be called *the central line of the strip.* By definition each strip contains a horizontal half-strip.

THEOREM XIX. (a) *If $F(s)$ is a holomorphic and bounded function in a curvilinear strip $\sum$ of width $2\pi a$ horizontal at the right and extending to $-\infty$ at the left,* (b) *if in the horizontal half-strip, contained in $\sum$, $F(s)$ is represented by $\sum a_n e^{-\lambda_n s}$ asymptotically with respect to an asymptotic sequence $\{\varphi_n(x)\}$,* (c) *if, on denoting by D the upper density of $\{\lambda_n\}$, $D<a$,* (d) *and if, on denoting by $\varphi(x)$ the envelope of $\{\varphi_n(x)\}$, by $\Lambda^*(u)$ the growth function of the excess distribution of $\{\lambda_n\}$, and by ω the quantity $\frac{1}{2(a-D)}$, one of the following conditions holds:*

(I) *there exists a positive constant p such that*

$$\Lambda^*(pu^\omega)\varphi(u) = O(1)$$

$$\liminf_{t=0+} \int_t^1 \log\left(\Lambda^*(pu^\omega)\varphi(u)\right) u^{\omega-1} du = -\infty,$$

(II) *there exists a constant $\omega' > \omega$ such that*

$$\int_0^1 \log \varphi(u) u^{\omega'-1} du = -\infty,$$

then $a_n = 0$ $(n \geqq 1)$.

It follows indeed from Theorem XV that the inequality (54) is satisfied for every $n \geqq 1$ with σ_1 negative numerically arbitrarily large. Thus, on fixing n in (54) and on making σ_1 tend to $-\infty$ we see that $a_n = 0$ $(n \geqq 1)$.

From this theorem, by means of Lemma VI, or directly from Theorem XVI (in the same manner as Theorem XIX follows from Theorem XV) follows the next theorem which concerns converging Dirichlet series.

THEOREM XX. *If $F(s)$ is holomorphic and bounded in a curvilinear strip of width $2\pi a$ horizontal at the right and extending to $-\infty$ at the left, and if this function is represented, for σ sufficiently large, by the series $\sum a_n e^{-\lambda_n s}$, the upper density of $\{\lambda_n\}$ satisfying the inequality $D<a$, then $F(s)$ is identically zero.*

Indeed $a_n = 0 (n \geqq 1)$, and, since $F(s) = \sum a_n e^{-\lambda_n s}$, we see that $F(s) \equiv 0$.

It should be remarked that from the hypotheses of Theorem XIX it also follows that $F(s) \equiv 0$ [**11**].

Each theorem which follows, and which is stated for converging Dirichlet series, is also true for series represented asymptotically with respect to an asymptotic sequence. We shall, however, not give those longer statements, since the reader will be able to form such statements, and to prove them by starting from Theorems XV or XIX in the same way as we do in starting from theorems on Dirichlet series and on basing ourselves on Theorems XVI or XX. We did, however, state Theorem XIX because it plays a very important rôle in the theory of quasi-analyticity (see [**11**]), and it could not be replaced, for that purpose, by the weaker theorems bearing on Dirichlet series. Theorem XIX can be considered as a far-going generalization of a theorem which gives a sufficient condition for the solution of a classical problem of Watson (see [**11, 13**]).

We shall suppose all through the remaining part of this course of lectures that the envisaged Dirichlet series *have an abscissa of convergence σ_C not equal to* ∞ : $\sigma_C < \infty$. We shall also denote systematically by D the upper density of $\{\lambda_n\}$.

An expression of the form $\sum_{n=1}^{m} a_n e^{-\lambda_n s}$ $(0 < \lambda_1 < \lambda_2 \cdots \lambda_m)$ shall be called *a Dirichlet polynomial* of degree λ_m.

Theorem XX can be generalized in the following manner:

THEOREM XXI. *If $F(s)$ is holomorphic in a curvilinear strip Σ of width $2\pi a$, horizontal at the right and extending to $-\infty$ at the left, if, for σ sufficiently large, $F(s)=\sum a_n e^{-\lambda_n s}$ with $D<a$, and if there exists a constant c such that $\lambda_m \leqq c < \lambda_{m+1}$ and such that in Σ, $e^{c\sigma}|F(s)| \leqq M < \infty$, then $F(s)$ is a Dirichlet polynomial of degree λ_m.*

It follows from the definition of c that $(\sum_{n=1}^{m} a_n e^{-\lambda_n s})e^{cs}$ is bounded in each half-plane $\sigma \leqq \sigma'$ (σ' arbitrary), and from the hypotheses of the theorem it follows then that the function

$$F_1(s) = \Big(F(s) - \sum_{n=1}^{m} a_n e^{-\lambda_n s}\Big)e^{cs}$$

is holomorphic and bounded in Σ, since, as is readily seen, for $\sigma > \sigma_c{}^F$, $(\sum_{m+1}^{\infty} a_n e^{-\lambda_n s})e^{cs}$ is also bounded. On the other hand, for $\sigma > \sigma_c{}^F$:

$$F_1(s) = \sum_{m+1}^{\infty} a_n e^{-(\lambda_n - c)s} = \sum_{1}^{\infty} d_n e^{-\mu_n s}$$

with $\mu_n = \lambda_{m+n} - c > 0$ ($n \geqq 1$). The function $F_1(s)$ satisfies the conditions of Theorem XX (in which $F(s)$ is replaced by $F_1(s)$) and therefore $F_1(s) \equiv 0$, which proves that

$$F(s) = \sum_{n=1}^{m} a_n e^{-\lambda_n s}.$$

THEOREM XXII. *If $F(s)$ is not a Dirichlet polynomial but is holomorphic in the part of the strip $\Sigma = \Sigma(s(u), \pi a)$ horizontal at the right and extending to $-\infty$ at the left, for which $\sigma \geqq \sigma_0$, and if, in this part of the strip, $F(s) = \sum a_n e^{-\lambda_n s}$, with $D < a$, then the analytic continuation of $F(s)$ through Σ satisfies one of the three conditions:*

a) *$F(s)$ admits in Σ at least one singularity.*

b) *For each positive constant c, $e^{c\sigma}|F(s)|$ tends to ∞ as σ tends to $-\infty$, uniformly with respect to t, $s = \sigma + it$ belonging to the strip $\Sigma(s(u), \pi(a-\epsilon))$, where $\epsilon > 0$ is arbitrary.*

c) *$F(s)$ takes in Σ each value, except at most one, infinitely many times.*

The proof of this theorem depends on certain facts of the theory of normal families of functions.

LEMMA IX. *Let $f(z, u)$ represent a holomorphic function of z in the circle $|z|<1$ for each value of u varying in an interval $[a, \infty)$. If the family composed of all the functions $f(z, u)$ is normal in $|z|<1$, and if, on denoting by $M(u, \alpha)$ the maximum of $|f(z, u)|$ in $|z| \leqq 1-\alpha$, where α is a fixed quantity $(0<\alpha<1)$, we have*

$$\lim_{u=\infty} M(u, \alpha) = \infty, \tag{88}$$

then

$$\lim_{u=\infty} f(z, u) = \infty, \tag{89}$$

uniformly in every circle $|z| \leqq 1-\delta$ $(0<\delta<1)$.

Suppose that the statement is not true. There exists then a quantity $0<\delta_1<1$ such that for a suitable sequence $\{u_i\}$ with $u_i \to \infty$, a sequence $\{z_i\}$ with $|z_i| \leqq 1-\delta_1$, and a constant $M<\infty$ the inequality $|f(z_i, u_i)|<M<\infty$ holds. But, since the family composed of the functions $f(z, u)$ is normal in $|z|<1$, it is possible to extract from the sequence $\{f(z, u_i)\}$ a subsequence $\{f(z, u_{i_n})\}$ which tends, in each circle $|z|<1-\eta$, $(0<\eta<1)$, uniformly either to a holomorphic function or to infinity. If we choose $\eta \leqq \min(\alpha, \delta_1)$ we see a contradiction, since by (88) this subsequence cannot tend to a holomorphic function in $|z|<1-\eta$, and from

$$|f(z_i, u_i)|<M<\infty$$

it follows that the limit cannot be infinity.

LEMMA X. [**12**] *Let $f(z, u)$ be a holomorphic function of z in $|z|<1$, for each $u \in [0, \infty)$. If in each circle*

$$|z| \leqq 1-\delta \ (0<\delta<1), \ \lim_{u=\infty} f(z, u) = \infty$$

uniformly, then there exists a positive function of $\delta: A(\delta) < \infty$ *such that*

$$\frac{1}{A(\delta)} < \frac{\log|f(z_1, u)|}{\log|f(z_2, u)|} < A(\delta), \tag{90}$$

for each couple of values z_1, z_2 *with* $|z_1| \leqq 1-\delta$, $|z_2| \leqq 1-\delta$ *and for* u *sufficiently large* $(u > u(\delta))$.

For u sufficiently large, $u > u_0$, $|f(z, u)| > 1$ for $|z| \leqq 1 - \frac{\delta}{2}$.

The family of harmonic functions of the variables (x, y), $(z = x + iy)$,

$$P_{\zeta, u}(x, y) = \frac{\log|f(z, u)|}{\log|f(\zeta, u)|},$$

where $|\zeta| \leqq 1-\delta$, $u > u_0$ (ζ and u are considered as parameters, each of which, when fixed, defines one of the functions of the family) is normal, since $P_{\zeta, u}(x, y) > 0$ [**15**]. But on setting $\zeta = \chi + i\eta$, we have

$$P_{\zeta, u}(\chi, \eta) = \frac{\log|f(\zeta, u)|}{\log|f(\zeta, u)|} = 1,$$

hence no subsequence extracted from a sequence of the family can tend, in $|z| \leqq 1-\delta$, uniformly to infinity. In other words, each such subsequence is bounded in such a circle. This proves that the whole family is bounded in $|z| \leqq 1 - \frac{3}{4}\delta$.

(This last statement is seen immediately, since, if the contrary were true, we should have a contradiction with the statement that each subsequence is bounded.) We have thus proved that

$$\frac{\log|f(z, u)|}{\log|f(\zeta, u)|} < A(\delta) \quad (|z| \leqq 1-\delta, \ |\zeta| \leqq 1-\delta).$$

On interchanging, in this inequality, z and ζ, and on putting $z = z_1$, $\zeta = z_2$, these inequalities give (90).

Let us now proceed to the proof of Theorem XXII. Sup-

pose that a) does not hold. We shall then have, by Theorem XVI, on setting $D<a'<a$:

$$|a_n| \leqq 2\pi a' \Lambda^*(a'-D) M(s(u)) \Lambda_n e^{\lambda_n \sigma(u)}, \tag{91}$$

for each $u \in (-\infty, \infty)$, where

$$M(s(u)) = \operatorname*{Max}_{s \in C(s(u),\, \pi a')} |F(s)|.$$

Since $F(s)$ is not a Dirichlet polynomial, there exists an infinite sequence $\{n_i\}$ such that $a_{n_i} \neq 0$. On fixing n_i, we see from (91) that, if $0<c<\lambda_{n_i}$:

$$\frac{|a_{n_i}| e^{(c-\lambda_{n_i})\sigma(u)}}{2\pi a' \Lambda^*(a'-D)\Lambda_{n_i}} \leqq M(s(u)) e^{\lambda_n c \sigma(u)}.$$

The left-hand expression tends (n_i fixed) to infinity, as u (and $\sigma(u)$) tends to $-\infty$. Thus $M(s(u)) e^{\lambda_n c \sigma(u)} \to \infty$ as $u \to -\infty$, for each $c>0$. Let us set

$$f(z, u) = F(s(-u) + \pi a z).$$

The functions $f(z, u)$ are holomorphic in $|z|<1$, and, on putting $\alpha = 1 - \dfrac{a'}{a}$, we see by the remarks just made that, if $M(u, \alpha) = \operatorname{Max} |f(z, u)|$, ($|z| \leqq 1-\alpha$), then, for each $c>0$:

$$\lim_{u=\infty} e^{\lambda_n c \sigma(-u)} M(u, \alpha) = \infty. \tag{92}$$

Let us now suppose that c) does not hold either. Then the family of functions $f(z, u)$ is normal in $|z|<1$, and we shall have, by Lemma IX, $\lim_{u=\infty} f(z, u) = \infty$, in every circle $|z| \leqq 1-\delta$ $(0<\delta<1)$. By Lemma X, (90) holds for each σ with $0<\delta<1$, z_1, z_2 being chosen arbitrarily in the circle $|z| \leqq 1-\delta$. Let us set $\delta<\alpha$. From (92) it follows that to each u there corresponds a quantity z_u, $|z_u| \leqq 1-\alpha<1-\delta$, such that for c arbitrary:

$$\lim_{u=\infty} (\lambda_n c \sigma(-u) + \log|f(z_u, u)|) = \infty,$$

(z_u is such that $\operatorname*{Max}_{|z| \leqq 1-\alpha} |f(z,u)| = M(u, \alpha)$, z_u is thus independent

of c). Since c is arbitrary we have also

$$\lim_{u=\infty} (\lambda_n c k(u)\sigma(-u) + \log|f(z_u, u)|) = \infty,$$

where $k(u)$ is any function such that $\frac{1}{A(\delta)} \leqq k(u) \leqq A(\delta)$. It follows then from (90), in which we set $z_1 = z_u$, $z_2 = z$, that, for $|z| \leqq 1-\delta$:

$$\lim_{u=\infty} (\lambda_n c\sigma(-u) + \log|f(z, u)|) = \infty,$$

that is to say, that, for $|s'| < \pi a(1-\delta)$:

$$\lim_{u=\infty} e^{\lambda_n c\sigma(-u)} F(s(-u) + s') = \infty.$$

On choosing $\delta = \frac{\epsilon}{a}$, this condition is equivalent to b). We have thus proved that if neither a) nor c) holds, then b) must hold, and the theorem is proved.

Theorem XXII was proved by the author [**10**]. In a weaker form it was proved in an earlier paper written in common with G. Gergen [**14**]. In this paper the authors considered strips of width $2\pi a$ with $a > \frac{1}{h}$, where

$$h = \lim_{n=\infty} \inf (\lambda_{n+1} - \lambda_n).$$

The Theorem XXII is more general since $hD \leqq 1$. On the other hand, it was supposed in this paper that $\sigma_c = -\infty$, and naturally the condition a) is eliminated *a priori*.

The question still remains open, whether or not the condition b) can really occur in the general case. We shall see, however, in the next chapter that, in a horizontal strip of which the width is determined by certain other considerations the condition b) certainly does not occur.

Remark. In the proof of Theorem XXII we have reasoned in the following fashion: on assuming the conditions of the theorem satisfied we have proved that if a) and c) do not hold, then the family of functions

$$f(z, u) = F(s(-u) + \pi a z)$$

considered as functions of z in $|z| < 1$ is *normal* and from that follows that b) must hold (by Lemmas IX and X). In other words, with the hypotheses of Theorem XXII one of the three following conditions is satisfied: a), b) or c'), which amounts to the property that the family of functions $F(s(-u) + \pi a z)$ (with the parameter $u \in (-\infty, \infty)$) is not normal in $|z| < 1$. But then this family admits in the unit circle an irregular point [**15**], that is to say, a point $z_0 (|z_0| < 1)$ such that in each neighborhood of this point the family is not normal and therefore in *each* such neighborhood of z_0 this family takes infinitely many times each value except at most one.

This remark will be useful in the next chapter.

VII

ENTIRE FUNCTIONS REPRESENTED BY DIRICHLET SERIES

We shall suppose in this chapter that $\sigma_C = -\infty$, in other words, that the series (1) converges for each value of s. It represents then an entire function $F(s)$, that is to say, a function holomorphic in the whole plane. Since $D < \infty$, we have also $\sigma_A = -\infty$, and the function is bounded on each straight line $\sigma = \sigma_1$.

Let us set

$$\mathcal{M}(\sigma) = \underset{-\infty < t < \infty}{\text{l. u. b.}} |F(\sigma + it)|.$$

J. Ritt [**18**] has introduced the notion of an order of an entire function represented by a Dirichlet series, which is different from the order of this function, when it is considered independently of the fact that it is given by a Dirichlet series. $\mathcal{M}(\sigma)$ being given as above, the quantity

$$\rho = \limsup_{\sigma = -\infty} \frac{\log_2 \mathcal{M}(\sigma)}{-\sigma} \tag{93}$$

shall be called *the order (R) of* $F(s)$. The order (R) of $F(s)$ is not to be confused with the order of $F(s)$ when it is considered merely as a function of s, without taking into special account the fact that it is given by a Dirichlet series. The order of $F(s)$ is, as is well known, the quantity

$$\delta = \limsup_{r = \infty} \frac{\log_2 M_1(r)}{\log r},$$

where $M_1(r) = \underset{|s| = r}{\text{Max}} |F(s)|$. For instance, if $F(s) = e^{-s}$ then

[1] $\log_2 a = \log(\log a)$.

$\rho=0$, $\delta=1$. Obviously if the order (R) of $\sum a_n e^{-n s}$ is ρ, then ρ is the order (in the classical sense, δ) of $\sum a_n z^n$. The following theorem is due to Ritt [**18**]:

THEOREM XXIII. *If* $F(s)=\sum a_n e^{-\lambda_n s}$, *with*

$$\sigma_A{}^F=-\infty, \liminf_{n=\infty}\frac{\lambda_n}{\log n}>0,$$

then a necessary and sufficient condition for $F(s)$ *to be of order* (R) *equal to* ρ *is that*

$$\limsup_{n=\infty}\frac{\log|a_n|}{\lambda_n\log\lambda_n}=-\frac{1}{\rho}, \tag{94}$$

$\left(\text{with } -\frac{1}{\rho}=-\infty \text{ if } \rho=0\right)$.

Let $F(s)$ be of a *finite* order (R) equal to ρ. Since for each σ:

$$|a_n|\leqq \mathcal{M}(\sigma)e^{\lambda_n\sigma},$$

and since, $\delta_1>0$ given, we have, by (93) for $-\sigma$ sufficiently large:

$$\log\mathcal{M}(\sigma)<e^{-(\rho+\delta_1)\sigma},$$

we see that, for $-\sigma$ sufficiently large:

$$\log|a_n|\leqq\log\mathcal{M}(\sigma)+\lambda_n\sigma<e^{-(\rho+\delta_1)\sigma}+\lambda_n\sigma.$$

Since the right-hand expression in this inequality takes its minimum at

$$\sigma=-(\rho+\delta_1)^{-1}\log\frac{\lambda_n}{\rho+\delta_1},$$

which tends to $-\infty$ as n tends to ∞, we see that, for n sufficiently large:

$$\log|a_n|\leqq\min_{-\infty<\sigma<\infty}(e^{-(\rho+\delta_1)\sigma}+\lambda_n\sigma)$$

$$=\frac{\lambda_n}{\rho+\delta_1}\left(1-\log\frac{\lambda_n}{\rho+\delta_1}\right),$$

which gives immediately:

$$\limsup_{n=\infty}\frac{\log|a_n|}{\lambda_n\log\lambda_n}\leqq-\frac{1}{\rho+\delta_1},$$

and since δ_1 is positive, arbitrary, it could be dropped in this inequality, that is to say:

$$\limsup_{n=\infty} \frac{\log|a_n|}{\lambda_n \log \lambda_n} \leqq -\frac{1}{\rho}. \tag{95}$$

If, on the other hand, this inequality holds with $\rho < \infty$, then to each $\delta_1 > 0$ there corresponds a quantity $B(\delta_1)$ such that $|a_n| < B(\delta_1) e^{-\frac{1}{\rho+\delta_1}\lambda_n \log \lambda_n} = B(\delta_1) \lambda_n^{-\frac{\lambda_n}{\rho+\delta_1}}$, and

$$\begin{aligned} \mathcal{M}(\sigma) &\leqq \sum |a_n| e^{-\lambda_n \sigma} \leqq B(\delta_1) \sum \lambda_n^{-\frac{\lambda_n}{\rho+\delta_1}} e^{-\lambda_n \sigma} \\ &\leqq B(\delta_1) \operatorname*{Max}_{n \geqq 1} e^{-\frac{1}{\rho+2\delta_1}\lambda_n \log \lambda_n - \lambda_n \sigma} \cdot \sum e^{-a\lambda_n \log \lambda_n}, \text{ where} \end{aligned} \tag{96}$$

$a = \dfrac{\delta_1}{(\rho+\delta_1)(\rho+2\delta_1)}$. But from $\liminf\limits_{n=\infty} \dfrac{\lambda_n}{\log n} > 0$, it follows that $\lambda_n > b \log n$ $(b > 0, n \geqq 1)$ and

$$\sum_2^\infty e^{-a\lambda_n \log \lambda_n} < \sum e^{-ab \log n \log \lambda_n} = \sum n^{-ab \log \lambda_n} < \infty.$$

Thus, by (96), for $-\sigma$ sufficiently large:

$$\begin{aligned} \mathcal{M}(\sigma) &\leqq C(\delta_1) \operatorname*{Max}_{n \geqq 1} e^{-\frac{1}{\rho+2\delta_1}\lambda_n \log \lambda_n - \lambda_n \sigma} \\ &\leqq C(\delta_1) \operatorname*{Max}_{x \geqq 0} e^{-\frac{x \log x}{\rho+2\delta_1} - x\sigma} = C(\delta_1)\, e^{-(k+2\sigma)}\, e^{-\left(1+\frac{\sigma}{k}\right)}, \end{aligned}$$

where $k = \dfrac{1}{\rho+2\delta_1}$. Thus for $-\sigma$ large:

$$\begin{aligned} \log \mathcal{M}(\sigma) &\leqq \log C(\delta_1) - (k+2\sigma) e^{-\left(1+\frac{\sigma}{k}\right)} \leqq e^{-\left(\frac{1}{k}+\delta_1\right)\sigma} \\ &= e^{-(\rho+3\delta_1)\sigma}, \end{aligned}$$

and

$$\limsup_{\sigma=-\infty} \frac{\log_2 \mathcal{M}(\sigma)}{-\sigma} \leqq \rho + 3\delta_1.$$

But since $\delta_1 > 0$ is arbitrary:

$$\limsup_{\sigma=-\infty} \frac{\log_2 \mathcal{M}(\sigma)}{-\sigma} \leqq \rho. \tag{97}$$

We have thus proved that from (95) with $\rho < \infty$ follows (97) and from (97) with $\rho < \infty$ follows (95). This is obviously

equivalent to our statement (even in the case where $\rho = \infty$).

We shall now introduce the notion of an order in a horizontal strip. Let $S(R)$ be a horizontal strip of width $2R$ that is to say, the set of points s with $|t-t_0| < R$, where t_0 is fixed. If $F(s)$ is holomorphic in $\bar{S}(R)$ ($S(R)$ closed) we shall set for each σ:

$$M_S(\sigma) = \operatorname*{Max}_{|t-t_0| \leq R} |F(\sigma+it)|,$$

and the quantity:

$$\limsup_{\sigma=-\infty} \frac{\log_2 M_S(\sigma)}{-\sigma} = \rho_S$$

shall be called *the order of $F(s)$ in the strip S.*

We shall prove the following theorem:

THEOREM XXIV. *If* $F(s) = \sum a_n e^{-\lambda_n s}$, *with*

$$\sigma_C{}^F = -\infty, \liminf (\lambda_{n+1} - \lambda_n) > 0,$$

and if the order (R) of $F(s)$ is positive, then the order of $F(s)$ in each horizontal strip $S(\pi a)$, with $a > D$, is equal to the order (R) of $F(s)$.

It follows from the hypotheses on $\{\lambda_n\}$ that

$$\liminf_{n=\infty} \frac{\lambda_n}{\log n} > 0,$$

and $D < \infty$.

If ρ is the order (R) of $F(s)$ in the plane, then, by Theorem XXIII, since the conditions of that theorem are satisfied:

$$\limsup_{n=\infty} \frac{\log|a_n|}{\lambda_n \log \lambda_n} = -\frac{1}{\rho}. \tag{98}$$

If $S(\pi a)$ is the strip $|t-t_0| < \pi a$, and if

$$M_1(\sigma_j + it_0) = \operatorname{Max} |F(s)|$$

when $s \in C(\sigma_j + it_0, \pi a)$, we see by Theorem XVI that, for σ_j arbitrary:

$$\log M_1(\sigma_j + it_0) \geqq \log|a_n| - \log \Lambda_n - \lambda_n \sigma_j - K, \ (n \geqq 1), \tag{99}$$

where K is a constant. By (98), $\delta_1 > 0$ given, there exists a sequence $\{n_j\}$ such that:

$$\log|a_{n_j}| > -\left(\frac{1}{\rho}+\delta_1\right)\lambda_n \log \lambda_{n_j}, \quad (j \geqq 1), \tag{100}$$

and by Theorem XVII:

$$\limsup_{n=\infty} \frac{\log \Lambda_n}{\lambda_n} < \infty,$$

that is to say:

$$\log \Lambda_n < P\lambda_n, \quad (n \geqq 1), \tag{101}$$

where P is a constant.

Let us denote by $s_j^* = \sigma_j^* + it_0^*$ a quantity such that $M_1(\sigma_j + it_0) = |F(s_j^*)|$. It is obvious that $M_S(\sigma_j^*) \geqq M_1(\sigma_j + it_0)$ and that $\sigma_j^* = \sigma_j + k_j$, where $|k_j| \leqq \pi a$. If we choose σ_j such as to have $\sigma_j = -\left(\frac{1}{\rho}+2\delta_1\right)\log \lambda_{n_j}$, where $\{n_j\}$ is a sequence such that (100) holds, the inequality (99) allows us to write

$$\log M_S(\sigma_j^*) = \log M_S(\sigma_j + k_j) = \log M_S\left(-\left(\frac{1}{\rho}+2\delta_1\right)\log \lambda_{n_j} + k_j\right)$$

$$\geqq \log M_1\left(-\left(\frac{1}{\rho}+2\delta_1\right)\log \lambda_{n_j} + it_0\right) \geqq \log|a_{n_j}| - \log \Lambda_{n_j}$$

$$+\left(\frac{1}{\rho}+2\delta_1\right)\lambda_{n_j}\log \lambda_{n_j} - K \geqq -\left(\frac{1}{\rho}+\delta_1\right)\lambda_{n_j} \log \lambda_{n_j} - P\lambda_{n_j}$$

$$+\left(\frac{1}{\rho}+2\delta_1\right)\lambda_{n_j} \log \lambda_{n_j} - K = \delta_1\lambda_{n_j} \log \lambda_{n_j} - P\lambda_{n_j} - K.$$

We have therefore:

$$\rho_S = \limsup_{\sigma=-\infty} \frac{\log_2 M_S(\sigma)}{-\sigma} \geqq \limsup_{j=\infty} \frac{\log_2 M_S(\sigma_j^*)}{-\sigma_j^*}$$

$$\geqq \limsup_{j=\infty} \frac{\log(\delta_1\lambda_{n_j} \log \lambda_{n_j} - P\lambda_{n_j} - K)}{\left(\frac{1}{\rho}+2\delta_1\right)\log \lambda_{n_j} - K_j} = \frac{1}{\frac{1}{\rho}+2\delta_1},$$

and, since $\delta_1 > 0$ is arbitrary, we have $\rho_S \geqq \rho$. But obviously $\rho_S \leqq \rho$, and therefore $\rho_S = \rho$.

This theorem was proved by the author [**10**] and, in weaker form, in the paper already mentioned above by

Gergen and the author [**14**]. There, as we said above, the rôle of D is played by $\frac{1}{h}$ where $h=\lim_{n=\infty}\inf(\lambda_{n+1}-\lambda_n)$, and, as we have said, $\frac{1}{h}\geqq D$.

It should be remarked that an order can also be defined in a curvilinear strip extending to $-\infty$ at the left. If $\Sigma=\Sigma(s(u), R)$ is such a strip, we shall set

$$M_\Sigma(\sigma)=\text{Max}|F(\sigma+it)|,$$

the maximum being taken when σ is fixed, the point $s+it$ belonging to Σ and $\rho_\Sigma=\limsup_{\sigma=-\infty}\frac{\log_2 M(\sigma)}{-\sigma}$. In the statement of Theorem XXIV "horizontal strip $S(\pi a)$" can be replaced by "curvilinear strip horizontal at the right and extending to $-\infty$ at the left, of width $2\pi a$," and there is nothing to change in the proof of Theorem XXIV in order to establish this new version of its statement.

G. Julia proved the following theorem [**15**]. If $f(z)$ is an entire function, there exists a straight line issuing from the origin such that in each angle with vertex at the origin, and containing this line, $f(s)$ takes infinitely many times each value except at most one. Such a line is called "*line of Julia*" or *line J*. For a Taylor-*D* series this can be translated in the following manner: If $\Sigma a_n e^{-n s}$ represents an entire function, $F(s)$, then in each horizontal (closed) strip of width 2π there exists a line $t=$const., such that in each horizontal strip containing this line $F(s)$ takes infinitely many times each value except at most one.

We shall generalize the notion as well as the theorem of Julia to general Dirichlet series. Let $F(s)$ be holomorphic in a horizontal strip S containing the line $t=t_0$. This line shall be called *line* $\overline{J}$, if in each horizontal strip containing

this line, and contained in S, $F(s)$ takes infinitely many times each value, except at most one.

We may define horizontal lines with even more specific properties than those of a line $\overline{J}$. Let us suppose, once more, that $F(s)$ is holomorphic in the horizontal strip $S(R)$ of width $2R$ defined by the inequality $|t-t_0|<R$, and let us consider the family $\mathfrak{F}$ of functions of z defined by $f(z,\sigma)=F(\sigma+it_0+z)$, $(-\infty<\sigma<\infty)$. Each of these functions is holomorphic in $C(0,R)$, that is to say, the circle $|z|<R$. If the family $\mathfrak{F}$ is not normal in $C(0,R)$ there exists, in $C(0,R)$, at least one irregular point [**15**], $z_0=x_0+iy_0$, that is to say, a point such that for no $\epsilon>0$ is the family $\mathfrak{F}$ normal in $C(z_0,\epsilon)\equiv(|z-z_0|<\epsilon)$. The straight line $t=t_0+y_0$ will be called a *line* J^0. Clearly a line J^0 *is a line* $\overline{J}$, since, if J^0 were not a line $\overline{J}$ there would exist an $\epsilon>0$ such that in $C(z_0,\epsilon)$ the functions of the family $\mathfrak{F}$ would take only a finite number of times two distinct values a and b $(a\neq b)$, and it would then be normal, contrary to the definition of a line J^0. In other words, a line J^0 has more precise properties than a line $\overline{J}$. A corresponding remark was already made with respect to lines J [**15**].

We shall now prove a theorem of a more specific nature than Theorem XXII, or even more specific than the statement which is contained in the remark at the end of Chapter VI. It is true that the conditions on $\{\lambda_n\}$ have to be those of Theorem XXIV, the function $F(s)$ has to be entire, and the strip horizontal, the width of which depends as well on D as on the order (R) of the function $F(s)$.

THEOREM XXV. *If* $F(s)=\sum a_n e^{-\lambda_n s}$ *with* $\sigma_c{}^F=-\infty$, $\liminf_{n=\infty}(\lambda_{n+1}-\lambda_n)>0$, *and if the order* (R) *of* $F(s)$ *is* $\rho>0$, *then in each horizontal strip* $S(\pi a)$ *of width* $2\pi a$, *with*

$a > \max\left(D, \frac{1}{2\rho}\right)$, *there exists a line* J^0 (*which is therefore a line* $\overline{J}$).

The proof of this theorem depends on Theorems XXII, XXIV, and on an interesting theorem of Valiron [**20**] which generalizes Picard's theorem to an angle. We shall state this result (here Lemma XI) for a horizontal strip.

Lemma XI. *If* $F(s)$ *is holomorphic in a horizontal strip containing a closed horizontal strip* $\overline{S} = \overline{S}\left(\frac{\pi}{\gamma}\right)$, *of width* $\frac{2\pi}{\gamma}$, *in which the order of* $F(s)$ *is larger than* $\frac{\gamma}{2}$, *then* $F(s)$ *takes in* $\overline{S}$ *infinitely many times each value, except at most one finite value.*

We shall only indicate that the proof of this lemma depends on the well known theorem of Schottky by which if a function $f(z) = c_0 + c_1 z + \cdots$ is holomorphic in $|z| < 1$ and does not take, in this circle, the two values 0 and 1, then there exists a function $K(u, v) < \infty$ such that, in

$$|z| < \theta \ (0 < \theta < 1), \ |f(z)| \leqq K(c_0, \theta).$$

A precision of this theorem, which is also needed for Valiron's proof, was given by Landau [**20**]: $K(u, v) < \frac{K_1(u)}{1-v}$, where $K_1(u) < \infty$ depends only on u.

In order to prove Theorem XXV it is then sufficient to make the following remarks: Suppose that the strip $S(\pi a)$ is given by $|t - t_0| < \pi a$, and let us choose $\epsilon > 0$ such that $a - \epsilon > \max\left(D, \frac{1}{2\rho}\right)$. By Theorem XXIV, the order ρ_{S_1} of $F(s)$ in the strip S_1 given by $|t - t_0| < \pi(a - \epsilon)$ is equal to ρ. On putting $\gamma = \frac{1}{a - \epsilon}$, we see that $\rho > \frac{\gamma}{2}$, and, by Lemma XI, $F(s)$ takes in S_1 infinitely many times each value except at

most one. The condition b) of Theorem XXII cannot therefore be satisfied in $S(\pi a)$, hence, by the remark at the end of Chapter VI, the family $\mathcal{F}$ of functions $f(z, \sigma) = F(\sigma + it_0 + z)$ is not normal in $|z| < \pi a$, and there exists an irregular point $z_0 = x_0 + iy_0$ with $|z_0| < \pi a$ (obviously, since $D < a$, the conditions of Theorem XXII are satisfied if the conditions of Theorem XXV are satisfied). The line given by $t = t_0 + y_0$ is a line J^0 which is situated in $S(\pi a)$.

For functions of positive order, Theorem XXV contains Julia's theorem on entire functions. Indeed if $F(s) = \sum a_n e^{-ns}$, $(\sigma_C{}^F = -\infty)$, is of order (R) equal to $\rho > 0$, since here $D \leqq 1$ we see that if k is an integer such that $k > \max\left(1, \frac{1}{2\rho}\right)$, $F(s)$ has a line $\overline{J}$ in the horizontal strip $|t| < k\pi$. But since $F(s)$ is periodic of period $2\pi i$, this function has a line $\overline{J}$ in the strip $|t| \leqq \pi$. That is to say, $\varphi(z) = \sum a_n z^n$ has a line J.

In one of his papers A. Bloch (see [**17**]) has suggested, without proof whatsoever, the following principle: if a property (P) bearing on sequences of complex numbers is such that from the fact that $\{a_n\}$ has this property and from $\limsup \sqrt[n]{|a_n|} = 1$ it follows that the point of affix $e^{i\theta}$ is a singularity for the series $\sum a_n z^n$, then from the fact that $\{b_n\}$ satisfies the property (P) and from $\lim \sqrt[n]{b_n} = 0$ should follow that the entire function given by $\sum b_n z^n$ admits the line of argument θ issuing from the origin as a line J.

This principle, necessarily somewhat vague as to the character of the property (P), was justified some years later by many interesting theorems. The first important contributions in that direction were made by G. Pólya [**17**]. Let $\{\lambda_n\}$ be a sequence of increasing positive integers. $N(x)$ being the distribution function of $\{\lambda_n\}$, Pólya introduced the notion of maximum density $\mathcal{D}$ which is defined in the following manner: let us set for $0 \leqq \xi \leqq 1$:

$$\mathcal{D}(\xi) = \limsup_{x=\infty} \frac{N(x) - N(\xi x)}{x(1-\xi)}.$$

This expression tends to a limit as ξ tends increasingly to 1, and

$$\mathcal{D} = \lim_{\xi=1-0} \mathcal{D}(\xi). \tag{102}$$

It is easily seen that $\mathcal{D} \geqq D$. Indeed, $\epsilon > 0$ given, from the definition of D it follows that there exists a sequence $\{x_i\}$ tending to infinity, such that $N(x_i) > (D-\epsilon)x_i (i \geqq 1)$ and such that for $0 < \xi < 1$, fixed, $N(\xi x_i) < (D+\epsilon)\xi x_i$, hence

$$\frac{N(x_i) - N(\xi x_i)}{x_i(1-\xi)} > D - \frac{\epsilon(1+\xi)}{1-\xi},$$

and

$$\mathcal{D}(\xi) \geqq \limsup_{i=\infty} \frac{N(x_i) - N(\xi x_i)}{x_i(1-\xi)} \geqq D - \frac{\epsilon(1+\xi)}{1-\xi}.$$

Since $\epsilon > 0$ is arbitrary we see that $\mathcal{D}(\xi) \geqq D$, and also $\mathcal{D} \geqq D$. It is obvious, since the λ_n are integers, that $\mathcal{D} \leqq 1$. Pólya proved that, if the Taylor series $\sum a_n z^{\lambda_n}$ (λ_n integers) represents an entire function $f(z)$ of order $\delta = \infty$, then $f(z)$ admits a line J in each angle with the vertex at the origin and of opening equal to $2\pi\mathcal{D}$.

Pólya's theorem is the first to justify Bloch's remark, since by a theorem also due to Pólya [**17**], if such a Taylor series instead of representing an entire function has a finite radius of convergence, equal say to unity, then it has a singularity on each arc of length $2\pi\mathcal{D}$ of the circle of convergence.

It is seen that Theorem XXV is more general than Pólya's theorem for the following reasons: in Theorem XXV, 1^0) a Dirichlet series is considered instead of a Taylor series; 2^0) the order is supposed to be only positive instead of being supposed to be equal to infinity; 3^0) if the order is equal to infinity Theorem XXV gives immediately the following one:

THEOREM XXVI. *If* $F(s)=\sum a_n e^{-\lambda_n s}$ *with* $\sigma_C{}^F=-\infty$, $\liminf_{n=\infty}(\lambda_{n+1}-\lambda_n)>0$, *and if the order* (R) *of* $F(s)$ *is* $\rho=\infty$, *then in each horizontal strip of width* $2\pi a$, *with* $a>D$, *there exists a line* J^0.

Thus, this statement, even if the λ_n are, moreover, supposed to be integers, is more general than Pólya's statement, since $\mathcal{D}\geqq D$.

Another remark which seems to be of a fundamental character can be made. For theorems concerning singularities the notion of maximum density, $\mathcal{D}$, seems to be inherent in the subject, as far as the study of singularities *on the axis of convergence* is concerned, since, for instance, in Pólya's theorem mentioned above, and concerning the singularities on the circle of convergence, the quantity $\mathcal{D}$ can certainly not be replaced by D. Therefore it seems that, even if a profound analogy does exist between the study of singularities and that of lines J (or $\overline{J}$), the second subject is of a more elementary character, at least the conditions required for the existence of lines J are simpler than those for the existence of singularities on the axis of convergence (or on segments of this axis of specified length).

As a corollary of Theorem XXVI let us state the following one.

THEOREM XXVII. *If* $F(s)=\sum a_n e^{-\lambda_n s}$ *with* $\sigma_C{}^F=-\infty$, $\liminf_{n=\infty}(\lambda_{n+1}-\lambda_n)>0$, $D=0$, *and if the order* (R) *of* $F(s)$ *is* $\rho=\infty$ *then each line* $t=t_0$ *with* t_0 *arbitrary, is a line* J^0.

This theorem was proved, for Taylor series, by Pólya. Let us finally mention that Theorem XXIV was also proved by Pólya for Taylor series, however, with the quantity $\mathcal{D}$ playing the rôle of D.

VIII

SINGULARITIES OF FUNCTIONS REPRESENTED BY DIRICHLET SERIES WITH FINITE UPPER DENSITY

The theory of detection of singularities of a function represented by a Dirichlet series has been very much enriched in the last twenty years. The subject is, however, masterfully treated in the book of V. Bernstein [**3**] which we have mentioned many times in this work, and we shall not attempt to give another systematic exposé of this matter. We shall therefore not give here the striking results of Pólya and V. Bernstein which are of a high degree of precision and generality (see [**3**]). Let us only recall that their theory is chiefly based on the notion of maximum density $\mathcal{D}$, due to Pólya and mentioned (in connection with Taylor series) above. Only results where the upper density is involved will be given in this chapter.

We shall now suppose that $-\infty < \sigma_C < \infty$. A *curvilinear channel connected with the half-plane* $\sigma > \sigma_C$ is a channel (as defined in Chapter IV) of which the central line contains at least one point situated in the half-plane $\sigma > \sigma_C$. To say that the function $F(s)$, represented by $\sum a_n e^{-\lambda_n s}$, admits in such a channel $\sum$ a singularity, means that there is no function holomorphic in this channel and which, for the points of $\sum$ for which $\sigma > \sigma_C$, is given by the sum of the Dirichlet series.

The sequences $\{\lambda_n\}$ and $\{a_n\}$ given, we shall set

$$\sigma(\{a_n\}, \{\lambda_n\}) = \limsup_{n=\infty} \frac{\log|a_n| - \log \Lambda_n}{\lambda_n},$$

where $\{\Lambda_n\}$ is the sequence associated with $\{\lambda_n\}$.

The following theorem can now be proved:

THEOREM XXVIII. *Suppose that $F(s)$ is represented by $\sum a_n e^{-\lambda_n s}$ with $\sigma_C{}^F < \infty$, $\sigma(\{a_n\}, \{\lambda_n\}) > -\infty$. $F(s)$ admits a singularity in every curvilinear channel of width $2\pi a$, with $a > D$, connected with the half-plane $\sigma > \sigma_C{}^F$, of which the central line contains at least one point $s_1 = \sigma_1 + it_1$ with*

$$\sigma_1 < \sigma(\{a_n\}, \{\lambda_n\}).$$

Let $\sum = \sum(s(u), \pi a)$ be a channel, connected with the half-plane $\sigma > \sigma_C{}^F$, of width $2\pi a$, the central line L of which contains a point $s_1 = s(u_1)$ with $\sigma_1 < \sigma(\{a_n\}, \{\lambda_n\})$, $(s_1 = \sigma_1 + it_1)$. If $F(s)$ did not have a singularity in $\sum$, $F(s)$ would be holomorphic and bounded in the channel $\sum(s(u), \pi a_1)$, where a_1 is such that $D < a_1 < a$. It would then follow from Theorem XVI that

$$\sigma_1 \geqq \limsup_{n=\infty} \frac{\log|a_n| - \log \Lambda_n}{\lambda_n} = \sigma(\{a_n\}, \{\lambda_n\}),$$

contrary to the supposition that $\sigma_1 < \sigma(\{a_n\}, \{\lambda_n\})$.

Let us now set

$$\sigma\{\lambda_n\} = \limsup_{n=\infty} \frac{\log \Lambda_n}{\lambda_n}.$$

It is obvious that:

$$\begin{aligned}\sigma(\{a_n\}, \{\lambda_n\}) &= \limsup_{n=\infty} \frac{\log|a_n| - \log \Lambda_n}{\lambda_n} \\ &\geqq \limsup \frac{\log|a_n|}{\lambda_n} - \limsup_{n=\infty} \frac{\log \Lambda_n}{\lambda_n} \\ &= \sigma_C - \sigma\{\lambda_n\}.\end{aligned}$$

It follows, on the other hand, from Theorem XVII that if $D < \infty$ and if (79) is satisfied, then

$$\sigma\{\lambda_n\} \leqq B(D, h),$$

with $B(D, h) = 3D(3 - \log(hD))$, if $D > 0$, and $B(0, h) = 0$.

The following theorem is therefore an immediate consequence of Theorem XXVIII.

THEOREM XXIX. *If $F(s)$ is represented by $\sum a_n e^{-\lambda_n s}$ with $\sigma_c^F > -\infty$ and if*

$$\liminf_{n=\infty}(\lambda_{n+1}-\lambda_n)=h>0,$$

then there exists a singularity of $F(s)$ in every curvilinear channel of width $2\pi a$ with $a>D$, connected with the half-plane $\sigma>\sigma_c^F$, the central line of which contains at least one point s_1 such that $\sigma_1<\sigma_c^F-B(D,h)$.

This theorem contains the following theorem due to A. Ostrowski [**16**].

THEOREM XXX. *There exists a positive function*

$$\alpha(h,D)(0<h<\infty,\ 0<D<\infty)$$

tending, when h is fixed, to zero as D tends to zero, and such that, if $F(s)$ is represented by $\sum a_n e^{-\lambda_n s}$ with $-\infty<\sigma_c$, the sequence $\{\lambda_n\}$ satisfying the condition

$$\liminf_{n=\infty}(\lambda_{n+1}-\lambda_n)=h>0,$$

and D being the upper density of $\{\lambda_n\}$, then $F(s)$ has at least one singularity in each circle of which the center is an arbitrary point of the axis of convergence and of which the radius is $\alpha(h,D)$.

More precisely $\alpha(h,D)=D[\pi+3(3-\log(hD))]$.

Since $\alpha(h,D)$ tends to zero with D (h being fixed), we have in particular the following theorem.

THEOREM XXXI. *If $F(s)$ is represented by $\sum a_n e^{-\lambda_n s}$ where $\sigma_c^F>-\infty$, and where the sequence $\{\lambda_n\}$ is such that*

$$\lim\frac{n}{\lambda_n}=0,^1$$

$$\liminf_{x=\infty}(\lambda_{n+1}-\lambda_n)>0,$$

then each point of the axis of convergence is a singularity for $F(s)$.

[1]This condition is obviously equivalent to $\lim_{x=\infty}\frac{N(x)}{x}=0$. Generally the two conditions $\limsup_{n=\infty}\frac{n}{\lambda_n}=D$, $\limsup_{x=\infty}\frac{N(x)}{x}=D$ are equivalent.

This theorem, when the λ_n are integers, is due to Fabry; the general case is due to Pólya [**17**].

Coming back to what we have said in Chapter VII about the analogy between the two kinds of theorems: those concerning the singularities of a Taylor series (or a Dirichlet series) and those concerning lines J (or lines $\overline{J}$, for Dirichlet series), we shall add the following remark which is a corollary of all the results established in both chapters VII and this one:

To theorems on lines $\overline{J}$ in a strip of width $2\pi a$ correspond theorems on singularities in such a strip and not theorems on singularities on a segment of the axis of convergence of length $2\pi a$.

IX

COMPOSITION THEOREMS OF HADAMARD TYPE

One of the most important theorems in the theory of the detection of singularities of a Taylor series, important in itself, as well as a very useful tool, is Hadamard's theorem on "the multiplication of singularities" discovered in 1898 (see for instance [6]).

Since we shall have later on in this chapter the opportunity to give this theorem in its most general form, we shall state it now in a form which is true only if the singularities are supposed to be isolated and not critical (not branch points), although many authors, by mistake, considered this statement as being the general one.

If E_1 is the set of the singularities of the function represented by $\sum a_n z^n$, E_2 the set of the singularities of the function represented by $\sum b_n z^n$, and E_3 the set of the singularities of the function represented by $\sum a_n b_n z^n$, then to each point e_3 of E_3 there correspond a point e_1 of E_1 and a point e_2 of E_2 such that $e_3 = e_1 e_2$.

Translated into Taylor-D series this theorem gives the following statement:

If γ is a singularity of the function represented by $\sum a_n b_n e^{-ns}$, there exists a singularity α of the function represented by $\sum a_n e^{-ns}$ and a singularity β of the function represented by $\sum b_n e^{-ns}$ such that $\gamma = \alpha + \beta$.

We have supposed in these statements that the radii of convergence of both series $\sum a_n z^n$, $\sum b_n z^n$ are finite, or, what amounts to the same thing, that the abscissa of convergence of the Taylor-D series is not $+\infty$.

The statement on Taylor-D series is not true for general Dirichlet series. In other words, the function represented by $\sum a_n b_n e^{-\lambda_n s}$ may have singularities, which are not of the form $\alpha+\beta$ where α is a singularity of the function $f(s)$ represented by $\sum a_n e^{-\lambda_n s}$ and β a singularity of the function $\varphi(s)$ represented by $\sum b_n e^{-\lambda_n s}$. This may occur even if each of the two functions $f(s)$ and $\varphi(s)$ has only one singularity. For instance, if $a_n = b_n = 1$, $\lambda_n = \log n$, then each of the functions $f(s)$, $\varphi(s)$ is the $\zeta(s)$-function of Riemann, which is known to have only one singularity, of affix one, which is a simple pole. Thus, if the theorem were true, the function represented by the composite series $(\sum a_n b_n e^{-\lambda_n s})$ should have as its only possible singularity the point of affix 2, but this composite function is, in the given circumstances, itself the function $\zeta(s)$.

The fact that the statement, given above for Taylor series, is not true for general Dirichlet series does not prove that another statement, which reduces to the one above when $\lambda_n = n$, may be true for the general case. We shall give here two theorems on Dirichlet series, one, of which the statement is relatively simple, and a second of a complex nature, but which deals with Dirichlet series of a very general type. Both statements contain Hadamard's theorem as a very particular case.

Although each of the theorems given below is true, with a certain precision of the language, for multiform functions we shall suppose, for the simplification of the language, that the functions represented by the given Dirichlet series are uniform. We shall see later on how these same statements can be modified if the corresponding functions are multiform.

Let $\sum a_n e^{-\lambda_n s}$ be a Dirichlet series with the abscissa of absolute convergence $\sigma_A < \infty$ and suppose that $-\infty < \sigma_1$.

Let Δ be a region, situated in the half-plane $\sigma > \sigma_1$, containing the part of this half-plane for which $\sigma > \sigma_A$ and such that there exists a function $f(s)$ holomorphic in Δ, equal to the sum of the series $\sum a_n e^{-\lambda_n s}$ for $s \in \Delta$, $\sigma > \sigma_A$. If there exists a domain $\Delta(\sigma_1)$ which contains each domain Δ having the properties mentioned, we shall say that the function $f(s)$, holomorphic in $\Delta(\sigma_1)$ and equal to the sum of the series for $s \in \Delta(\sigma_1)$, $\sigma > \sigma_A$, is uniform in the half-plane $\sigma > \sigma_1$. The set composed of all the points of the half-plane $\sigma \geqq \sigma_1$ which are not points of $\Delta(\sigma_1)$ will be denoted by $S_f^{\sigma_1}$ and called the *singular set of* $f(s)$ *with respect to the half-plane* $\sigma > \sigma_1$. The points s such that $\sigma = \sigma_1$ thus always belong to $S_f^{\sigma_1}$. If $\Delta(\sigma_1)$ coincides with the half-plane $\sigma > \sigma_1$, only the points with $\sigma = \sigma_1$ belong to $S_f^{\sigma_1}$. Interesting theorems will be obtained only if $S_f^{\sigma_1}$ contains points different from those situated on the line $\sigma = \sigma_1$, but for the general validity of the results to come it is important to define $S_f^{\sigma_1}$ as we did. When σ_1 is given, the function defined as above in $\Delta(\sigma_1)$ will be denoted by $f(s) = \sum a_n e^{-\lambda_n s}$, although $f(s)$ is actually equal to the sum of the series only in the part of $\Delta(\sigma_1)$ for which $\sigma > \sigma_C^f$. Since, if $\sigma_1' < \sigma_1$, the function defined, as indicated above, by means of the analytical continuation of $\sum a_n e^{-\lambda_n s}$ in the regions $\Delta(\sigma_1')$ and $\Delta(\sigma_1)$ (if it is uniform in both regions) takes, in the intersection of these regions, the same values, the indication of this function by the same letter f is justified. It should be remarked that $\Delta(\sigma_1)\Delta \subset (\sigma_1')$. Thus, to say that "$f(s) = \sum a_n e^{-\lambda_n s}$ is uniform in the half-plane $\sigma > \sigma_1$" means that $\Delta(\sigma_1)$ exists, that $f(s)$ is holomorphic in $\Delta(\sigma_1)$, that this function is given in the part of $\Delta(\sigma_1)$ in which $\sigma > \sigma_C^f$ by the sum of the series $\sum a_n e^{-\lambda_n s}$, and that $f(s)$ cannot be continued analytically to a point of $S_f^{\sigma_1}$ without effecting this continuation on a path crossing the line $\sigma = \sigma_1$.

It is well known that a function represented by a general Dirichlet series does not have necessarily a singularity on the axis of convergence, contrary to what happens in the case of a Taylor-*D* Series. For instance, the function $f(s)$ represented by the series $\sum \frac{(-1)^n}{n^s}$ is an entire function and yet here $\sigma_C^f = 0$ $(\sigma_A^f = 1)$. But, let H_f be the greatest lower bound of all the quantities σ' such that $f(s)$ is holomorphic if $\sigma > \sigma'$ and is given in the part of this half-plane in which $\sigma > \sigma_C^f$ by $\sum a_n e^{-\lambda_n s}$. This quantity H_f is called the *abscissa of holomorphism of* $f(s) = \sum a_n e^{-\lambda_n s}$, [3]. Obviously

$$H_f \leqq \sigma_C^f \leqq \sigma_A^f.$$

The line represented by $\sigma = H_f$ is called the axis of holomorphism of the series. The quantity

$$H_f(\sigma_1) = \text{l. u. b.}_{s \in S_f^{\sigma_1}} \sigma$$

shall be called the *abscissa of holomorphism of* $f(s) = \sum a_n e^{-\lambda_n s}$ *in the half-plane* $\sigma > \sigma_1$. Obviously $H_f(\sigma_1) = H_f$ if $\sigma_1 \leqq H_f$, and $H_f(\sigma_1) = \sigma_1$ if $\sigma_1 > H_f$.

To say that the *only possible singularities of* $\sum a_n e^{-\lambda_n s}$ *in the half-plane* $\sigma > \sigma_1$ *are points of the closed set* E will mean that, on denoting by $P(\sigma_1)$ this half-plane and by $(P(\sigma_1) - E)$ the part of the difference of sets $P(\sigma_1) - E$ which constitutes a region[1] containing a half-plane $\sigma > \sigma_2$ in which $\sum a_n e^{-\lambda_n s}$ converges, the function $f(s)$ is holomorphic in $(P(\sigma_1) - E)$ and is given in the part of this region for which $\sigma > \sigma_2$ by the sum of the series.

It should be remarked that, if $\sigma_1' > \sigma_1$, $S_f^{\sigma_1'}$ does not necessarily contain all the points of $S_f^{\sigma_1}$. This is the reason why we use, in the definition of $S_f^{\sigma_1}$, the expression "with respect to the half-plane $\sigma > \sigma_1$," instead of "in the half-plane $\sigma > \sigma_1$."

[1] A region is a *connected* set of interior points.

Let $f(s)=\sum a_n e^{-\lambda_n s}$ be uniform in the half-plane $\sigma>\sigma_1$, and let us set

$$S_f^{\sigma_1}(\epsilon)=\bigcup_{s\in S_f^{\sigma_1}} C(s,\epsilon),^1$$

where $C(s,\epsilon)$ is the open circle of center s and radius ϵ. If to every $\epsilon>0$ there corresponds a constant $K(\epsilon)(=K(\epsilon,\sigma_1))$ such that in the closed region $\Delta(\sigma_1,\epsilon)=\overline{(P(\sigma_1)-S_f^{\sigma_1}(\epsilon))},^2$ $|f(s)|<K(\epsilon)$, we shall say that $f(s)$ is *bounded in the half-plane* $\sigma>\sigma_1$, *except for singularities*, or "bounded e. f. s." It is obvious that each Taylor-D series, $f(s)=\sum a_n e^{-ns}$, which is uniform in the half-plane $\sigma>\sigma_1$, is bounded e. f. s. in this half-plane. This is, however, not true for general Dirichlet series. For instance, $\zeta(s)=\sum\frac{1}{n^s}$ is not bounded e. f. s. in the half-plane $\sigma>1-\delta$, if $\delta>0$.

Suppose now that the function $f(s)=\sum a_n e^{-\lambda_n s}$, still uniform in the half-plane $\sigma>\sigma_1$, is such that there exists a non-negative constant m having the following property: to each $\epsilon>0$ there corresponds a constant $K(\epsilon,m)(=K(\epsilon,m,\sigma_1))$ such that in $\Delta(\sigma_1,\epsilon)$: $|f(s)|<K(\epsilon,m)|t|^m$, if $|t|$ is sufficiently large. Let μ be the greatest lower bound of such quantities m. Thus to each $\delta>0$ and each $\epsilon>0$ there corresponds a constant $N(\epsilon,\delta)(=N(\epsilon,\delta,\sigma_1))$ such that

$$|f(s)|<N(\epsilon,\delta)|t|^{\mu+\delta}$$

in $\Delta(\sigma_1,\epsilon)$, for $|t|$ sufficiently large, and no quantity smaller than μ has this property. We shall say that $f(s)$ is *of order* μ *except for singularities*, or of order μ e. f. s., *in the half-plane* $\sigma>\sigma_1$.

Let us denote by $Q(\sigma_1,t_0)$ the "quarter-plane" given by $\sigma>\sigma_1$, $t>t_0$. Suppose that the series $\sum a_n e^{-\lambda_n s}$ has the fol-

[1] $\bigcup L(a)$ means the union of all the sets $L(a)$ (depending on the parameter a) $a\in A$ as a takes all the possible values in the set A.

[2] $\Delta(\sigma_1,\epsilon)$ is therefore also connected, closed, and contains a half-plane $\sigma>\sigma_2$.

lowing properties: let a region Δ and a quarter-plane $Q(\sigma_2, t_0)$ be such that $Q(\sigma_1, t_0) \supset \Delta \supset Q(\sigma_2, t_0)$ and such that there exists a function $f(s)$ holomorphic in Δ, given in $Q(\sigma_2, t_0)$, by the sum of the series; we shall suppose that there exists a region $D(\sigma_1, t_0)$ which contains all the regions Δ having the properties just described. We shall then say that $f(s) = \sum a_n e^{-\lambda_n s}$ is *uniform in the "quarter-plane"* $Q(\sigma_1, t_0)$. We shall denote by $S_f^{\sigma_1, t_0}$ the set of all the points s satisfying the inequalities $\sigma \geqq \sigma_1$, $t > t_0$ which are not points of $D(\sigma_1, t_0)$. The set $S_f^{\sigma_1, t_0}$ shall be called *the singular set of* $f(s)$ *with respect to the quarter-plane* $Q(\sigma_1, t_0)$. We shall set

$$S_f^{\sigma_1, t_0}(\epsilon) = \bigcup_{s \in S_f^{\sigma_1, t_0}} C(s, \epsilon).$$

If to every $\epsilon > 0$ there corresponds a constant

$$K(\epsilon) \left(= K(\epsilon, \sigma_1, t_0)\right)$$

such that in the closed region $\Delta(\sigma_1, t_0, \epsilon) = \overline{\left(Q(\sigma_1, t_0) - S_f^{\sigma_1, t_0}(\epsilon)\right)}$, which is a part of the closure of $Q(\sigma_1, t_0) - S_f^{\sigma_1, t_0}(\epsilon)$ connected with a quarter-plane $Q(\sigma_2, t_1)$, we have $|f(s)| < K(\epsilon)$, we shall say that $f(s)$ is *bounded in* $Q(\sigma_1, t_0)$ *except for singularities* (e. f. s.). Obviously if $f(s)$ is uniform and bounded e. f. s. in the half-plane $\sigma > \sigma_1$, it is also uniform and bounded e. f. s. in every quarter-plane $Q(\sigma_1, t_0)$ with t_0 arbitrary. For each t_0 such that $f(s)$ is holomorphic, uniform, and bounded in $Q(\sigma_1, t_0)$, (σ_1 fixed), let us set

$$\sigma_1^*(t_0) = \text{l. u. b.}_{s \in S_f^{\sigma_1, t_0}} \sigma,$$

and let us also set

$$H_f^+(\sigma_1) = \lim_{t_0 = \infty} \sigma_1^*(t_0).$$

Obviously $H_f^+(\sigma_1)$ exists and $H_f^+(\sigma_1) \geqq \sigma_1$. This quantity $H_f^+(\sigma_1)$ shall be called the *ultimate* (+) *abscissa of holomorphism of* $f(s)$ *in the half-plane* $\sigma > \sigma_1$.

If A and B are two sets of complex numbers, we shall call the set composed of all the numbers $\alpha + \beta$, where $\alpha \in A$, $\beta \in B$, the *sum-composite* set of the sets A and B.

It should be remarked that, even if both sets A and B are closed sets, their sum-composite is not necessarily closed.

Since the first of the theorems given in this chapter bears on functions represented by Dirichlet series which are bounded e. f. s. either in a half-plane or in a quarter-plane, it is important to show that Dirichlet series, which are not Taylor-D series (nor series of the form $\sum a_n e^{-k n s}$ with k constant), exist which have such properties. Let us suppose that $f(s) = \sum a_n e^{-\lambda_n s}$ with $\sigma_A^f < 0$, the λ_n being arbitrary, and let $T(z) = \sum c_n z^n$ be an entire function. Consider the function

$$F(s) = T\left(f(s) + \frac{1}{1-e^{-s}}\right) = T\left(\sum a_n e^{-\lambda_n s} + \sum e^{-ns}\right)$$
$$= \sum_m c_m \left(\sum_n a_n e^{-\lambda_n s} + \sum_n e^{-ns}\right)^m.$$

Since for $\sigma > 0$ the series $\sum_m |c_m| \left(\sum_n |a_n| e^{-\lambda_n \sigma} + \sum_n e^{-n\sigma}\right)^m$ converges, we see that $F(s)$ is represented for $\sigma > 0$ by a Dirichlet series

$$F(s) = \sum d_n e^{-\nu_n s},$$

where each ν_n is of the form $\nu_k = \sum_1^p \alpha_i \lambda_i + \beta$, where the α_i and β are non-negative integers. On the other hand, in the half-plane $\sigma > \sigma_A^f$ the function $f(s) + \frac{1}{1-e^{-s}}$ has only simple poles of affixes $2k\pi i$. In the region $\sigma > \sigma_A^f + \epsilon$, outside the circles $C(2k\pi i, \epsilon)$, this function is bounded. Therefore $F(s)$ is bounded there. But inside each such circle the values taken by $f(s) + \frac{1}{1-e^{-s}}$ cover a neighborhood of infinity, and by Picard's theorem $F(s)$ takes in these circles each value except at most one, infinitely many times. Thus, the points of affixes $2k\pi i$ are isolated essential singularities. $F(s)$ has therefore the required properties.

The theorem we shall now prove contains Hadamard's

theorem on Taylor series, but it deals with far less general series than Theorem XXIII.

THEOREM XXXII. *Suppose that* $f(s)=\sum a_n e^{-\lambda_n s}$, *with* $\sigma_A{}^f<\infty$, *is uniform and bounded e. f. s. in the quarter-plane* $\sigma>\sigma_1$, $t>t_0$, *and suppose that* $\varphi(s)=\sum b_n e^{-\lambda_n s}$ *with* $\sigma_A{}^\varphi<\infty$, *is uniform and bounded e. f. s. in the half-plane* $\sigma>\sigma_2$. *Let* $S_f{}^{\sigma_1, t_0}$ *be the singular set of* $f(s)$ *with respect to the quarter-plane* $\sigma>\sigma_1$, $t>t_0$, *and* $S_\varphi{}^{\sigma_2}$ *the singular set of* $\varphi(s)$ *with respect to the half-plane*: $\sigma>\sigma_2$. *Let* $H_f{}^+(\sigma_1)$ *be the ultimate* $(+)$ *abscissa of holomorphism of* $f(s)$ *in* $\sigma>\sigma_1$ *and* $H_\varphi(\sigma_2)$ *the abscissa of holomorphism of* $\varphi(s)$ *in* $\sigma>\sigma_2$.

The series $\sum a_n b_n e^{-\lambda_n s}$ *has an abscissa of absolute convergence not greater than* $\sigma_A{}^f+\sigma_A{}^\varphi$, *the function* $F(s)=\sum a_n b_n e^{-\lambda_n s}$ *is uniform in the half-plane* $\sigma>H_f{}^+(\sigma_1)+\sigma_2$ *and the only possible singularities of* $F(s)$ *in this half-plane are the points of the sum-composite set of the sets* $S_f{}^{\sigma_1, t_0}$, *the limit-points of this set, and the points of* $S_\varphi{}^{\sigma_2}$.

Let c and t_1 be chosen such that $c>\sigma_A{}^f$, $t_1>t_0$ and consider

$$F_T(z)=\frac{1}{iT}\int_{c+it_1}^{c+iT} f(s)\varphi(z-s)ds, \quad (T>t_1), \tag{103}$$

the path of integration being on the line $\sigma=c$. $F_T(z)$ is holomorphic if $x=\mathcal{R}(z)>c+\sigma_A{}^\varphi$, since then $z-s$, s varying on the path of integration, lies in the half-plane $\sigma>\sigma_A{}^\varphi$. For such a value of z we have

$$F_T(z)=\frac{1}{iT}\int_{c+it_1}^{c+iT} f(s)\varphi(z-s)ds$$

$$=\frac{1}{iT}\int_{c+it_1}^{c+iT} f(s)\sum b_n e^{-\lambda_n(z-s)}ds=\sum b_n e^{-\lambda_n z}\cdot\frac{1}{iT}\int_{c+it_1}^{c+iT} f(s)e^{\lambda_n s}ds.$$

But by Theorem IX:

$$\frac{1}{iT}\int_{c+it_1}^{c+iT} f(s)e^{\lambda_n s}ds=a_n+\epsilon_n(T), \tag{104}$$

with $\lim_{T=\infty}\epsilon_n(T)=0$. Therefore

$$F_T(z)=\sum a_n b_n e^{-\lambda_n z}+\sum \epsilon_n(T)b_n e^{-\lambda_n z}.$$

The series $\sum a_n b_n e^{-\lambda_n z}$ converges absolutely for $x > \sigma_A^{\varphi} + c$, since

$$\sum_{p+1}^{\infty} |a_n b_n| e^{-\lambda_n x} = \sum_{p+1}^{\infty} |b_n| e^{-\lambda_n x} \left| \lim_{T=\infty} \frac{1}{iT} \int_{c+il_1}^{c+iT} f(s) e^{\lambda_n s} ds \right|$$

$$\leqq \mathcal{M}(c) \sum_{p+1}^{\infty} |b_n| e^{-\lambda_n (x-c)},$$

where

$$\mathcal{M}(c) = \underset{-\infty < t < \infty}{\text{l. u. b.}} |f(c+it)| \leqq \sum |a_n| e^{-\lambda_n c},$$

and $\sum |b_n| e^{-\lambda_n (x-c)} < \infty$ for $x - c > \sigma_A^{\varphi}$. Thus $\sum a_n b_n e^{-\lambda_n s}$ converges absolutely for $\sigma \geqq \sigma_A^{f} + \sigma_A^{\varphi}$.

For $x > \sigma_A^{\varphi} + c$ we have then, for each p:

$$|F_T(z) - F(z)| = \left| \sum_1^{\infty} b_n e^{-\lambda_n z} (a_n + \epsilon_n(T)) - \sum_1^{\infty} a_n b_n e^{-\lambda_n z} \right|$$

$$\leqq \sum_1^{\infty} |b_n| |\epsilon_n(T)| e^{-\lambda_n x}.$$

It follows from the definition of $\epsilon_n(T)$ that (see (104)):

$$|\epsilon_n(T)| < 2\mathcal{M}(c) e^{\lambda_n c},$$

hence, for each p:

$$|F_T(z) - F(z)| \leqq \sum_1^{p} |b_n| |\epsilon_n(T)| e^{-\lambda_n x} + 2\mathcal{M}(c) \sum_{p+1}^{\infty} |b_n| e^{-\lambda_n (x-c)},$$

which proves, since (when p is fixed) $\epsilon_n(T) \to 0$ as $T \to \infty$ $(n = 1, \cdots p)$, and $\sum_{p+1}^{\infty} |b_n| e^{-\lambda_n (x-c)} \to 0$ with $p \to \infty$, $(x - c > \sigma_A^{\varphi})$, that $\lim_{T=\infty} F_T(z) = F(z)$, as $x > \sigma_A^{\varphi} + c$.

Let us now suppose that $c > \max(\sigma_A^{f}, \sigma_1)$ and let us pave the strip $\sigma_1 \leqq \sigma \leqq c$ with squares of side $\epsilon = \frac{c - \sigma_1}{q}$ (q integer), and let $\mathcal{D}(\epsilon)$ be the set composed of all the squares which neither contain (in their closure) a point of $S_f^{\sigma_1, t_0}$, nor are contiguous to such a square. If ϵ is sufficiently small (if q is sufficiently large), all the squares bordering the line $\sigma = c$ belong to $\mathcal{D}(\epsilon)$. We shall suppose that ϵ is chosen in

such a way. Let then $\mathcal{D}^*(\epsilon)$ be the greatest region belonging to $\mathcal{D}(\epsilon)$ and containing the squares bordering the line $\sigma = c$, and $\mathcal{D}^*(\epsilon, t_1)$ be the part of $\mathcal{D}^*(\epsilon)$ for which $t > t_1$. By the properties of $f(s)$ this function is holomorphic and bounded in $\mathcal{D}^*(\epsilon, t_1)$. Let $C(\epsilon)$ be the boundary of $\mathcal{D}^*(\epsilon, t_1)$. Let $\{\alpha_m\}$ be a sequence of quantities increasing to infinity with m, such that $\alpha_m > t_1$, $(m \geqq 1)$, and let us denote by $\mathcal{D}^*(\epsilon, t_1, m)$ the part of $\mathcal{D}^*(\epsilon, t_1)$ for which $t < \alpha_m$, and by $C(\epsilon, t_1, m)$ the part of its frontier which does not contain the points of $\sigma = c$. From Cauchy's theorem it follows that, if $x > \sigma_A^{\varphi} + c$ (the integral being taken in an appropriate sense):

$$F_{\alpha_m}(z) = \frac{1}{i\alpha_m} \int_{C(\epsilon, t_1, m)} f(s)\varphi(z-s)ds, \tag{105}$$

and thus, still for $x > \sigma_A^{\varphi} + c$:

$$F(z) = \lim_{m=\infty} \frac{1}{i\alpha_m} \int_{C(\epsilon, t_1, m)} f(s)\varphi(z-s)ds.$$

Let l_m be the set of the points forming the segments of $C(\epsilon, t_1, m)$ for which $t = \alpha_m$, and L the set of the points forming the segments of $C(\epsilon, t_1, m)$ for which $t = t_1$. Since the total length of each of those sets of segments for a given m, and that of L, is not larger than $c - \sigma_1$, and since on these segments $|f(s)|$ and $|\varphi(z-s)|$ (if $x > \sigma_A^{\varphi} + c$) are bounded, we see that, on denoting by $C'(\epsilon, t_1, m)$ the set $C(\epsilon, m) - l_m - L$, we have also:

$$F(z) = \lim_{m=\infty} \frac{1}{i\alpha_m} \int_{C'(\epsilon, t_1, m)} f(s)\varphi(z-s)ds, \tag{106}$$

where t_1 is fixed and satisfies only the inequality $t_1 > t_0$.

Let now Δ be a bounded region in the z-plane such that it contains points with $x > \sigma_A^{\varphi} + c$, such that for each $z \in \Delta$: $x > H_f^+(\sigma_1) + \sigma_2 - 3\epsilon$, and such that, on denoting by $S_{f,\varphi}$ the sum-composite of $S_f^{\sigma_1, t_0}$ and $S_{\varphi}^{\sigma_2}$, the inequality $|z - \gamma| > 6\epsilon$ holds for each $z \in \Delta$ and each $s \in S_{f,\varphi}$. If t_1 is chosen suffi-

ciently large, then for each $s \in C(\epsilon, t_1, m)$, $z \in \Delta$, $\beta \in S_\varphi^{\sigma_2}$ the inequalities

$$\mathcal{R}(z-s) = x-\sigma > \sigma_2, \quad |z-s-\beta| > \epsilon$$

are satisfied. Indeed, if t_1 is sufficiently large, we have for each $s \in C(\epsilon, t_1, m)$: $\sigma < H_f^+(\sigma_1) + 3\epsilon$, and, if $z \in \Delta$: $x-\sigma > \sigma_2$; on the other hand, if for a $z' \in \Delta$, an $s' \in C(\epsilon, t_1, m)$ and a $\beta' \in S_\varphi^{\sigma_2}$ we had had $|z'-s'-\beta'| < \epsilon$, since to each s' of $C(\epsilon, t_1, m)$ there corresponds an $\alpha' \in S_f^{\sigma_1, t_0}$ such that $|s'-\alpha'| < 2\sqrt{2}\epsilon$, we would have $|z'-\alpha'-\beta'| \leqq |z'-s'-\beta'| + |s'-\alpha'| < (2\sqrt{2}+1)\epsilon$, contrary to the supposition that $|z'-\alpha'-\beta'| > 6\epsilon$. Thus, with our choice of t_1 if $s \in C(\epsilon, t_1, m)$, $z \in \Delta$, the variable $u = z-s$ lies in the part of the half-plane $\sigma > \sigma_2$ which is connected with the half-plane $\sigma > \sigma_A^\varphi$, and in which the function $\varphi(s)$ is holomorphic, uniform, and bounded. The functions $F_{\alpha_m}(z)$ defined by (105) constitute thus a bounded family of holomorphic functions in Δ. But in a closed part of Δ situated in the half-plane $x > \sigma_A^\varphi + c$, the series $F_{\alpha_m}(z)$ converges uniformly towards $F(z) = \sum a_n b_n e^{-\lambda_n z}$, and by a classical theorem of Stieltjes-Vitali [**15**] $F_{\alpha_m}(z)$ converges uniformly to a holomorphic function in each closed part of Δ. This limit is the analytic continuation of $F(z)$. We have therefore proved that the series $\sum a_n b_n e^{-\lambda_n z}$ can be continued analytically in each region Δ described above, and this proves the theorem.

This theorem was proved by the author in 1929 [**8**], in a different wording and in a somewhat more general form. In 1940 S. Bochner [**4**] proved a very similar theorem in which the sets $S_f^{\sigma_1, t_0}$, S_φ, $S_{f,\varphi}$ are replaced by the boundaries of the stars of the corresponding functions.[1] The boundary of a star of a function $f(s)$ represented by $\sum a_n e^{-\lambda_n s}$ is the

[1] S. Bochner mentions in his paper Theorem XXXIII which was reproduced in Bernstein's book [3], but he does not seem to have had knowledge of Theorem XXXII. It should be remarked that Bochner states his theorem also for general almost-periodic functions.

set of the points composed of all the half-lines $t=t^*$, $\sigma \leqq \sigma^*$, which have the following property: in the region remaining after the extraction of these half-lines the function represented by $\sum a_n e^{-\lambda_n s}$ can be continued analytically, this being not true if one of the half-lines is replaced by $t=t^*$, $\sigma<\sigma_1^*$ with $\sigma_1^*<\sigma^*$. There is nothing to change in the proof of Theorem XXXII if in its statements the sets $S_f^{\sigma_1, t_0}$, $S_\varphi^{\sigma_2}$ are, respectively, replaced by $E_f^{\sigma_1, t_0}$, $E_\varphi^{\sigma_2}$ where $E_f^{\sigma_1, t_0}$ is the set composed of the part of the boundary of the star of $f(s)$ situated in the quarter-plane $\sigma>\sigma_1$, $t>t_0$ and the half-line $\sigma=\sigma_1$, $t>t_0$, and where $E_\varphi^{\sigma_2}$ is the boundary of the part of the star of $\varphi(s)$ situated in the half-plane $\sigma>\sigma_2$. The introduction of the stars allows us to avoid the introduction of the uniformity of the function. It should, however, be remarked that, on introducing stars (at least for uniform functions), the region in which it can be asserted that the composite function is holomorphic is diminished. On the other hand, in order to be able to state Theorem XXXII as well, for multiform functions as for uniform ones, curvilinear stars can be introduced. In other words, the set $E_\varphi^{\sigma_2}$, for instance, can be replaced by a set of simple Jordan arcs which connect the line $\sigma=\sigma_2$ to different points in the half-plane $\sigma>\sigma_2$ and which are such that in the remaining part of this half-plane $\sum a_n e^{-\lambda_n s}$ can be continued analytically. The reader will see for himself how $E_f^{\sigma_1, t_0}$ could be conveniently replaced. The arcs, for each function $f(s)$ and $\varphi(s)$, respectively, should be chosen in such a way that they do not intersect each other, except if one of these arcs is a part of another. The fact that these arcs can be taken in a very arbitrary way allows the determination of a very general region of holomorphism for the composite function.

Theorem XXXII contains Hadamard's theorem, since for Taylor-D series σ_1 and σ_2 can be taken arbitrarily (if the

functions are uniform; for multiform functions the introduction of the arcs described above is essential). The fact that t_0 of the statement can be taken arbitrarily does not improve the theorem for the Taylor-D case, since the functions $f(s)$ and $\varphi(s)$ are periodic with period $2\pi i$. But the possibility of an arbitrary choice of t_0 does play an important rôle in the applications of Theorem XXXII [**5**]. It should be remarked that for two Taylor-D series the sum-composite of $S_f^{\sigma_1}$ and $S_\varphi^{\sigma_2}$ is closed, since the points of both sets $S_f^{\sigma_1}$ and $S_\varphi^{\sigma_2}$ are distributed periodically with period $2\pi i$.

Before we begin to deal with Dirichlet series of a very general character, let us give the following definition. Let $\{\mu_n\}$ be a sequence of positive quantities strictly increasing to infinity, let $f(s)=\sum a_n e^{-\lambda_n s}$, and let k be a positive integer. Let n_0 be the smallest integer n such that $\mu_n > \lambda_1$, and let us set:

$$a_n^{(k)} = 0 \ (0 < n < n_0) \ (\text{if } n_0 > 1)$$
$$a_n^{(k)} = \sum_{\lambda_m < \mu_n} (\mu_n - \lambda_m)^k a_m, \ (n \geqq n_0).$$

The quantity $a_n^{(k)}$ shall be called the n^{th} *coefficient of* $f(s)$ *with respect to the sequence* $\{\mu_n\}$, *of order* k.

THEOREM XXXIII. *Suppose that* $f(s)=\sum a_n e^{-\lambda_n s}$ *with* $\sigma_A^f < \infty$, *is uniform and of order* ν *e. f. s., in the half-plane* $\sigma > \sigma_1$, *and that* $\varphi(s)=\sum b_n e^{-\mu_n s}$, *with* $\sigma_A^\varphi < \infty$, *is uniform and of order* μ *e. f. s. in the half-plane* $\sigma > \sigma_2$.

Let $S_f^{\sigma_1}$ *be the singular set of* $f(s)$ *with respect to the half-plane* $\sigma > \sigma_1$, *and* $S_\varphi^{\sigma_2}$ *the singular set of* $\varphi(s)$ *with respect to the half-plane* $\sigma > \sigma_2$. *Let* $H_f(\sigma_1)$ *be the abscissa of holomorphism of* $f(s)$ *in* $\sigma > \sigma_1$.

If k *is an integer such that* $k > \nu + \mu$, *the series* $\sum a_n^{(k)} b_n e^{-\mu_n s}$, *where* $a_n^{(k)}$ *is the* n^{th} *coefficient of* $f(s)$ *with respect to the sequence* $\{\mu_n\}$ *of order* k, *has an abscissa of absolute convergence not larger than* $\max(\sigma_A^\varphi, \sigma_A^f + \sigma_A^\varphi)$, *the function* $F_k(s)=\sum a_n^{(k)} b_n e^{-\mu_n s}$

is uniform in the half-plane $\sigma > \mathrm{Max}(H_f(\sigma_1), 0) + \sigma_2$, *and the only possible singularities of* $F_k(s)$ *in this half-plane are the points of the sum-composite of the sets* $S_f^{\sigma_1}$ *and* $S_\varphi^{\sigma_2}$, *the limit-points of this set, and the points of* $S_\varphi^{\sigma_2}$.

Let $c > \max(0, \sigma_A^f)$, and consider:

$$F(z) = \frac{k!}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{f(s)\varphi(z-s)}{s^{k+1}} ds, \tag{107}$$

the path of integration being the line $\sigma = c$. $F(z)$ is holomorphic if $x = \mathcal{R}(z) > c + \sigma_A^\varphi$, since then $z-s$, s varying on the path of integration, lies in the half-plane $\sigma > \sigma_A^\varphi$. For such values of z we may write:

$$\begin{aligned} &\frac{k!}{2\pi} \int_{c-i\infty}^{c+i\infty} f(s) \sum b_n e^{-\mu_n(z-s)} \frac{ds}{s^{k+1}} \\ &= \sum b_n e^{-\mu_n z} \cdot \frac{k!}{2\pi} \int_{c-i\infty}^{c+i\infty} f(s) e^{\mu_n s} \frac{ds}{s^{k+1}} \\ &= \sum_n b_n e^{-\mu_n z} \frac{k!}{2\pi} \int_{c-i\infty}^{c+i\infty} \sum_m a_m e^{(\mu_n - \lambda_m)s} \frac{ds}{s^{k+1}}, \end{aligned} \tag{108}$$

and by Theorem X

$$F(z) = \sum_{n=n_0}^{\infty} b_n e^{-\mu_n z} \sum_{\lambda_m < \mu_n} a_m (\mu_n - \lambda_m)^k = \sum_1^\infty a_n^{(k)} b_n e^{-\mu_n z}, \tag{109}$$

n_0 being the smallest integer n such that there exists a quantity λ_m smaller than μ_n.

Obviously the last series converges for $x > c + \sigma_A^\varphi$. But in (108) all the equalities hold if $f(s)$ is replaced by $\sum |a_n| e^{-\lambda_n s}$, $\varphi(s)$ by $\sum |b_n| e^{-\mu_n s}$.

Therefore the series

$$\sum_{n=n_0}^{\infty} \sum_{\lambda_m < \mu_n} |a_m| (\mu_n - \lambda_m)^k b_n e^{-\mu_n x}$$

converges for $x > c + \sigma_A^\varphi$, and so does the series $\sum |a_n^{(k)} b_n| e^{-\mu_n x}$, which proves that the abscissa of absolute convergence of $\sum a_n^{(k)} b_n e^{-\lambda_n z}$ is not greater than $\max(\sigma_A^\varphi, \sigma_A^f + \sigma_A^\varphi)$, since c has only to be such that

$$c > \max(0, \sigma_A^f).$$

The equality (109) proves that $F(z) = F_k(z)$.

Let us now set, moreover, $c > \max(0, \sigma_1, \sigma_A^f)$, and, as for the proof of Theorem XXXII, let us pave the strip $\sigma_1 \leqq \sigma \leqq c$ with squares of side $\epsilon = \frac{c-\sigma_1}{q}$ (q integer). Let $\mathcal{D}_1(\epsilon)$ be the set composed of all the squares which contain (in their closure) neither a point of $S_f^{\sigma_1}$ nor the point $s=0$, and which are not contiguous to such squares.

If ϵ is sufficiently small, all the squares bordering on the line $\sigma = c$ will belong to $\mathcal{D}_1(\epsilon)$. Let us choose ϵ so that this is the case, and let $\mathcal{D}_1^*(\epsilon)$ be the greatest region belonging to $\mathcal{D}_1(\epsilon)$ and containing the squares bordering the line $\sigma = c$. The function $f(s)$ is holomorphic in $\mathcal{D}_1^*(\epsilon)$, and to each $\delta_1 > 0$ there corresponds a constant $N_1(\delta_1)(= N_1(\delta_1, \epsilon, \sigma_1))$ such that, in $\mathcal{D}_1^*(\epsilon)$, $|f(s)| < N_1(\delta_1)|t|^{\mu+\delta_1}$, for $|t|$ sufficiently large.

On denoting by $\mathcal{D}_1^*(\epsilon, T)$ the set of points of $\mathcal{D}_1^*(\epsilon)$ for which $|t| < T$, let $C_1^*(\epsilon, T)$ be the part of its boundary which does not contain the points of $\sigma = c$. For $x > \sigma_A^\varphi + c$ we have, by Cauchy's theorem

$$\int_{C_1^*(\epsilon, T)} f(s)\varphi(z-s)\frac{ds}{s^{k+1}} = \int_{c-iT}^{c+iT} f(s)\varphi(z-s)\frac{ds}{s^{k+1}},$$

the first integral being taken, in an appropriate sense, on $C_1^*(\epsilon, T)$, the second on the straight-line segment joining the two points $c-iT$ and $c+iT$. If L^+ and L^- are the set of segments of $C_1^*(\epsilon, T)$ on which we have, respectively, $t = T$, $t = -T$, we see immediately that the integrals extended over these sets of segments tend to zero as $T \to \infty$. This proves that, for $x > \sigma_A^\varphi + c$:

$$F_k(z) = \frac{k!}{2\pi i}\int_{C_1^*(\epsilon)} f(s)\varphi(z-s)\frac{ds}{s^{k+1}}, \tag{110}$$

where $C_1^*(\epsilon)$ is the part of the boundary of $\mathcal{D}_1^*(\epsilon)$ which does not contain the points of $\sigma = c$.

Let now Δ be a bounded region of the z-plane, such that it contains points with $x > \sigma_A^{\varphi} + c$, such that for each $z \in \Delta$: $x > \max(H_f(\sigma_1), 0) + \sigma_2 + 2\epsilon$, and such that, on denoting by $S_f^*{}_{\varphi}$ the union of the sum-composite of the two sets $S_f^{\sigma_1}$, $S_{\varphi}^{\sigma_2}$ and of the set $S_{\varphi}^{\sigma_2}$ ($S_f^*{}_{\varphi}$ is therefore the sum-composite of two sets, one of which is composed of the points of $S_f^{\sigma_1}$ and the point $s=0$, the other being the set $S_{\varphi}^{\sigma_2}$), the inequality $|z-\gamma| > 6\epsilon$ holds for each $z \in \Delta$, and each $\gamma \in S_f^*{}_{\varphi}$. The inequalities $\mathcal{R}(z-s) = x-\sigma > \sigma_2$, $|z-s-\beta| > \epsilon$ are then satisfied for each $z \in \Delta$, $s \in C_1^*(\epsilon)$, and $\beta \in S_{\varphi}^{\sigma_2}$. This can be proved in a manner analogous to that in which we have proved a similar fact in proving Theorem XXXII. Thus, if $s \in C_1^*(\epsilon)$, $z \in \Delta$, the variable $u = z-s$ lies in the part of the half-plane $\sigma > \sigma_2$ which is connected with the half-plane $\sigma > \sigma_A^{\varphi}$ and in which the function $\varphi(s)$ is holomorphic, uniform, and is such that to each $\delta_2 > 0$ there corresponds a constant $N_2(\delta_2)$ $(= N_2(\delta_2, \epsilon, \sigma_2))$ with the property that $|\varphi(s)| < N_2(\delta_2)|t|^{\nu+\delta_2}$, for $|t|$ sufficiently large. Since the region Δ is bounded, we see thus that, if $s \in C_1^*(\epsilon)$, $z \in \Delta$, then $|\varphi(z-s)| < N_3(\delta_2)|t|^{\nu+\delta_2}$ for $|t|$ sufficiently large. Since $\nu + \mu < k$, the constants δ_1 and δ_2 can be chosen in such a manner as to have $\mu + \nu + \delta_1 + \delta_2 < k$, and then the integral (110) converges uniformly with respect to z as z varies in any fixed closed region $\bar{\Delta}_1 \subset \Delta$. Therefore $F_k(z)$ is holomorphic in Δ. This obviously proves our theorem.

This theorem was proved by the author, in the same paper as Theorem XXXII [**8**] in a somewhat more general and precise form. The statement given here is somewhat easier to prove than the original one. Some simplifications of the details are taken from the proof of the author's theorem given by V. Bernstein [**3**]. Let us remark that, in using the integral (110), introduced by the author, D. Widder proved a particular case of Theorem XXXIII [**21**].

Hadamard's theorem is contained in Theorem XXX, at least when the Taylor series in question are supposed to represent uniform functions. Let indeed

$$\psi(s) = \sum_{2}^{\infty} d_n e^{-ns}, \quad \varphi(s) = \sum_{1}^{\infty} b_n e^{-ns},$$

and let $\alpha = e^{s_0}$ be one of the singularities of $\psi(s)$ on the axis of convergence. $\psi_1(s) = \sum d_n \alpha^n e^{-ns} = \sum A_n e^{-ns}$ admits the point $s=0$ as a singularity. Let us now set $f(s) = (1-e^{-s})^2 e^s \psi_1(s)$ $= (1-e^{-s})^2 e^s \sum_{2} A_n e^{-ns} = \sum_{1} a_n e^{-ns}$. We have, as is readily seen,

$$A_n = (n-1)a_1 + (n-2)a_2 + \cdots + a_{n-1} = \sum_{m<n} (n-m)a_m, \ (n \geqq 2).$$

The singularities of $f(s)$ are those of $\psi_1(s)$ except perhaps the points $2k\pi i$, which are singular for $\psi_1(s)$, but may be regular for $f(s)$. Since for the Taylor-D series, $f(s)$, $\varphi(s)$, the quantities σ_1 and σ_2 can be taken as negative numbers, arbitrarily large in absolute value, and since, with any such choice of σ_1 and σ_2, k can be taken equal to unity, we see, by Theorem XXX, that the only possible singularities of $F(s) = \sum a_n^{(1)} b_n e^{-ns}$, where $a_n^{(1)} = \sum_{m<n} a_m(n-m) = A_n \ (n>1)$, $a_1^{(1)} = 0$, are the points of the sum-composite $S_{f\varphi}$ of the sets S_f and S_φ and the points of S_φ, where S_f is the set of singularities of $f(s)$ and S_φ the set of singularities of $\varphi(s)$,[1] in the whole plane. If we denote by S_{ψ_1} the set of singularities of $\psi_1(s)$, since this function admits as singularities all the points of S_f and the point $s=0$, we see that the sum-composite of S_{ψ_1} and S_φ is the union of the two sets $S_{f\varphi}$ and S_φ. In other words, the only possible singularities of $F(s) = \sum A_n b_n e^{-ns} = \sum d_n b_n e^{ns_0} e^{-ns}$ are the points of the sum-composite of the sets S_{ψ_1} and S_φ. On denoting by S_ψ the set of singularities of $\psi(s)$, it is obvious that each point of

[1]For a uniform function the set of singularities is, by the definition we have adopted in various circumstances, the complement to the region of existence of the function (and not only the boundary of this region).

S_ψ is obtained by adding $-s_0$ to an affix of a point of S_{ψ_1}. Therefore, the only possible singularities of $\Phi(s)=\sum d_n b_n e^{-ns}$ are those of the sum-composite of the sets S_ψ and S_φ. This constitutes the most precise form of Hadamard's theorem, at least when the functions are supposed to be uniform. If the functions are multiform, the statement of Theorem XXX can easily be adapted, on introducing, as we have said above in speaking of Theorem XXXII, simple arcs of Jordan which lead to uniform functions.

Returning to Theorem XXX, we notice that if $\sigma_1<0$, and if $s=0$ does not belong to $S_f^{\sigma_1}$, then for ϵ sufficiently small the boundary $C_1^*(\epsilon)$ contains a square (of side 3ϵ) which contains inside the point $s=0$, this square being separated from the other points of $C_1^*(\epsilon)$. Let $c(\epsilon)$ be the boundary of this square. The part of $F_k(z)$ given by the integration on $c(\epsilon)$ furnishes, for x sufficiently large (by the theorem of residues):

$$
\begin{aligned}
(111)\qquad I_k(z) &= \frac{k!}{2\pi i}\int_{c(\epsilon)} f(s)\varphi(z-s)\frac{ds}{s^{k+1}} = \frac{\partial^k[f(s)\varphi(z-s)]}{\partial s^k}_{(s=0)} \\
&= (-1)^k[f(0)\varphi^{(k)}(z) - C_k^1 f'(0)\varphi^{(k-1)}(z) \\
&\qquad + \cdots + (-1)^k f^{(k)}(0)\varphi(z)].
\end{aligned}
$$

On the other hand, it follows readily from the proof of Theorem XXX that the only singularities of the function

$$F_k(z) - I_k(z) = \frac{k!}{2\pi i}\int_{C_2^*(\epsilon)} f(s)\varphi(z-s)\frac{ds}{s^{k+1}},$$

where $C_2^*(\epsilon)=C_1^*(\epsilon)-c(\epsilon)$, are the points of the sum-composite of $S_f^{\sigma_1}$ and $S_\varphi^{\sigma_2}$. From the form of $I_k(z)$ it follows readily that the only singularities of this function are those of $\varphi(z)$. Theorem XXXIII can therefore be stated in the following more precise form:

THEOREM XXXIV. *With the same hypotheses as in Theorem XXXIII, if $s=0$ is a regular point for $f(s)$, the function $F_k(s)=\sum a_n^{(k)} b_n e^{-\mu_n s}$ is uniform in the half-plane $\sigma>H_f(\sigma_1)+\sigma_2$,*

and the only possible singularities, in this half-plane, of the function

$$F_k(s) - I_k(s),$$

where

$$I_k(s) = (-1)^k [f(0)\varphi^{(k)}(s) - C_k^1 f'(0)\varphi^{(k-1)}(s) + \cdots + (-1)^k f^{(k)}(0)\varphi(s)],$$

are the points of the sum-composite set of the sets $S_f^{\sigma_1}$ *and* $S_\varphi^{\sigma_2}$ *and their limit points.*

The fact that in the general Theorem XXXIII (or Theorem XXXIV), the two sequences $\{\lambda_n\}$, $\{\mu_n\}$ can be chosen as entirely unrelated sequences allows us a great variety of interesting applications. For instance, the distribution of singularities of functions represented by Taylor series is much better known than the distribution of singularities of functions represented by general Dirichlet series. Real progress would therefore be achieved if the problem of the location of singularities of a Dirichlet series could be reduced to the problem of the location of singularities of a relatively simple Dirichlet series and that of a Taylor series, of which the coefficients are simply related to the sequences of the exponents and coefficients of the given Dirichlet series.

We shall show that in many cases an indication of the location of singularities of $\sum a_n e^{-\lambda_n s}$ can be obtained when the affixes of the singularities of $\sum e^{-\lambda_n s}$ and those of a Taylor series of which the coefficients are formed with the quantities a_n and λ_n are known.

Let $\{\lambda_n\}$ be a sequence having the usual properties and, moreover, such that $\lambda_1 > 1$, let $\{a_n\}$ be a sequence of complex numbers, and let $k > 0$ be an integer. If there exists a sequence $\{d_n\}$ such that for each $n \geqq 1$:

$$a_n = \sum_{m<\lambda_n} (\lambda_n - m)^k d_m,$$

we shall say that $\{a_n\}$ admits a *generatrix-sequence* $\{d_n\}$ *of order k, with respect to* $\{\lambda_n\}$.

Generally a sequence $\{a_n\}$ has no generatrix-sequence of a given order, with respect to a given sequence $\{\lambda_n\}$. But conditions bearing on $\{\lambda_n\}$ can easily be given in order that each sequence $\{a_n\}$ will have a generatrix-sequence of a given order, with respect to $\{\lambda_n\}$. Let us denote by e_n the greatest integer smaller than λ_n. The following lemma is immediate.

LEMMA XII. *If for an infinite sequence* $\{n_i\}$, $n_i < \lambda_{n_i} \leqq n_i + 1$ *and if the determinant* Δ_{n_i} *of order* n_i*—of which the term in the* p^{th} *row and the* q^{th} *column is* $(\lambda_p - q)^k$ *(if* $1 \leqq p \leqq n_i$, $q \leqq e_p$*) and* 0 *(if* $1 \leqq p < n_i$, $e_p < q \leqq e_{n_i} = n_i$*)—is different from zero, then each sequence* $\{a_n\}$ *has a generatrix-sequence of order* k *with respect to* $\{\lambda_n\}$.

Indeed, each system of equations (with the d_n as unknowns),

$$\sum_{m<\lambda_1} (\lambda_1 - m)^k d_m = a_1$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\sum_{m \leqq n_i} (\lambda_{n_i} - m)^k d_m = a_{n_i},$$

admits a system of solutions. If n_{i_1} and n_{i_2} belong both to the sequence $\{n_i\}$ and if $n_{i_1} < n_{i_2}$, the quantities $d_1^{(i_2)}$, $d_2^{(i_2)}$, $\cdots$ $d_{n_{i_1}}^{(i_2)}$, which are the first n_{i_1} terms of the solution of the system (S_{i_2}), constitute also the system of solutions of the system (S_{i_1}). The union of the solutions of all the systems (S_i), $(i \geqq 1)$, constitutes therefore a sequence $\{d_n\}$ (to each n there corresponds only one d_n), which satisfy all the systems of equations:

$$\sum_{m<\lambda_1} (\lambda_1 - m)^k d_m = a_1$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \qquad (n \geqq 1)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\sum_{m<\lambda_n} (\lambda_n - m)^k d_m = a_n.$$

LEMMA XIII. *If* $n<\lambda_n\leqq n+1$ $(n\geqq 1)$, *each sequence* $\{a_n\}$ *has a generatrix-sequence of any order* $k>0$, *with respect to* $\{\lambda_n\}$.

Here $n_i=i$, and obviously

$$\Delta=[(\lambda_1-1)(\lambda_2-2)\cdots(\lambda_n-n)]^k\neq 0;$$

the conclusion follows then from Lemma XII.

From Theorem XXXIII follows immediately [7]:

THEOREM XXXV. *Let* $F(s)$ *be represented by* $\sum a_n e^{-\lambda_n s}$, $(\lambda_1>1)$. *Suppose that* $\varphi(s)=\sum e^{-\lambda_n s}$ *is uniform and of order* $\mu<\infty$ *e.f.s. in a half-plane* $\sigma>\sigma_2$, *and that* $\{a_n\}$ *admits a generatrix-sequence* $\{d_n\}$ *of an integral order* $k>\mu$, *with respect to* $\{\lambda_n\}$. *If*

$$\limsup_{n=\infty}\frac{\log|d_n|}{n}=a<\infty,$$

and if $f(s)=\sum d_n e^{-ns}$ *is uniform, then* $F(s)$ *is uniform and the only possible singularities of* $F(s)$ *in the half-plane* $\sigma>\mathrm{Max}(a,0)+\sigma_2$ *are the points of the sum-composite of* S_f *and* $S_\varphi^{\sigma_2}$, *their limit-points, and the points of* $S_\varphi^{\sigma_2}$, *where* S_f *and* $S_\varphi^{\sigma_2}$ *are, respectively, the set of singularities of* $f(s)$ *in the whole plane and the singular set of* $\varphi(s)$ *with respect to the half-plane* $\sigma>\sigma_2$.

This is a particular case of Theorem XXXIII with $\{d_n\}$ playing the rôle of $\{a_n\}$ of Theorem XXXIII, the b_n of that theorem being here all equal to unity, σ_1 being an arbitrary quantity, $H_f(\sigma_1)=a$ for $\sigma_1<a$, a_n of the statement of Theorem XXXV being nothing but the n^{th} coefficient of $f(s)=\sum d_n e^{-ns}$ with respect to the sequence $\{\lambda_n\}$, of order k.

It is seen by Theorem XXXV that each time that the sequence $\{\lambda_n\}$ satisfies the conditions of Lemma XII (or, in particular, of Lemma XIII) then only the behavior of the function $\varphi(s)=\sum e^{-\lambda_n s}$ has to be known, so far as Dirichlet series, different from Taylor-D series, are concerned.

REMARK. If the sequence $\{\lambda_n\}$ is such that to each posi-

tive integer k there corresponds at most one λ_n, such that $k<\lambda_n \leqq k+1$, in other words, with the notations given above, if $(e_{n+1}-e_n) \geqq 1$, $(n \geqq 1)$, (with $\lambda_1 > 1$), then the series $\sum a_n e^{-\lambda_n s}$ can be written in the form $\sum a'_p e^{-L_p s}$, where $L_{e_n}=\lambda_n$, and where L_p for p distinct from each e_n is chosen arbitrarily in the interval $(p, p+1]$, where $a'_{e_n}=a_n$ and where $a'_p=0$, for p distinct from each e_n. The quantities L_p are now such that $p<L_p \leqq p+1$. Thus, by Theorem XXXV, the study of the singularities of $F(s)=\sum a_n e^{-\lambda_n s}$ can still be reduced to that of the singularities of a function represented by a Dirichlet series of which all the coefficients are equal to unity, $\sum e^{-L_n s}$, and a Taylor-D series $\sum d'_n e^{-ns}$, where $\{d'_n\}$ is the generatrix-sequence of $\{a'_n\}$, with respect to $\{L_n\}$. The order of the generatrix-sequence depends, of course, on the order e. f. s. of $\varphi_1(s)=\sum e^{-L_n s}$ (if this function is of finite order e. f. s. in a certain half-plane $\sigma > \sigma_2$).

It is of course important to give composition theorems, in which the order e. f. s. of the composed functions is not involved, even if these functions are not bounded e. f. s. in certain half-planes. This can be achieved if the sets $S_f^{\sigma_1}$, $S_\varphi^{\sigma_2}$ are replaced by sets which contain, respectively, the sets $S_f^{\sigma_1}$ and $S_\varphi^{\sigma_2}$ but which for Taylor-D series are reduced to the sets $S_f^{\sigma_1}$, $S_\varphi^{\sigma_2}$. We shall give here an example of such a theorem, proved by the author [7]. But first let us make the following remarks: Let $f(s)=\sum a_n e^{-\lambda_n s}$ be uniform in a half-plane $\sigma > \sigma_1$. We shall denote by $E_f(a, \sigma_1)$ the set of points in the half-plane $\sigma > \sigma_1$ in which $f(s)$ takes the value a. In other words, if $\rho \in E_f(a, \sigma_1)$, then $\mathcal{R}(\rho) > \sigma_1$ and $f(\rho)=a$. Let now $\varphi(s)=\sum b_n e^{-\mu_n s}$ be uniform in the half-plane $\sigma > \sigma_2$, and let, as usual, $S_\varphi^{\sigma_2}$ be the singular set of $\varphi(s)$ with respect to the half-plane $\sigma > \sigma_2$. We shall denote by $P[f, \sigma_1; \varphi, \sigma_2]$ the set of points having the following property: a necessary and sufficient condition that $\gamma \in P[f, \sigma_1; \varphi, \sigma_2]$ is that if a

takes any complex value, except at most one, to every $\epsilon>0$ there corresponds a point $\rho_a=\rho_a(\epsilon)\in E_f(a,\sigma_1)$, and a point $\beta=\beta(\epsilon)\in S_\varphi^{\sigma_2}$ such that $|\gamma-\rho_a-\beta|<\epsilon$. If both series, the series representing $f(s)$ and that representing $\varphi(s)$, are Taylor-D series, then the set $P[f,\sigma_1;\ \varphi,\sigma_2]$ is a subset of the sum-composite of the sets $S_f^{\sigma_1}$ and $S_\varphi^{\sigma_2}$. Indeed, let $\gamma\in P[f,\sigma_1;\ \varphi,\sigma_2]$, and let $\beta_n\in S_\varphi^{\sigma_2}$, $\rho_{\omega_n}\in E_f(\omega_n,\sigma_1)$, with $\omega_n\to\infty$ as $n\to\infty$, be such that $\lim_{n=\infty}(\rho_{\omega_n}+\beta_n)=\gamma$, which is possible by the definition of γ. Since in our case $f(s)$ and $\varphi(s)$ are periodic with period $2\pi i$, on setting $\rho_{\omega_n}=\rho'_{\omega_n}+i\rho''_{\omega_n}$, we can suppose that $0\leqq\rho''_{\omega_n}\leqq 2\pi$, and then there exist two integers k_1, k_2 such that, on setting $\beta_n=\beta'_n+i\beta''_n$, $2k_1\pi\leqq\beta''_n\leqq 2k_2\pi$, all these equalities and inequalities hold for $n\geqq 1$. On the other hand, let σ' be such that $\sigma'>\sigma_A^f$; if for $n>n_0$ $|\omega_n|>\operatorname{Max}_{\sigma=\sigma'}|f(s)|$, then all the $\rho_{\omega_n}(n>n_0)$, are situated in the rectangle Δ, defined by $\sigma_1\leqq\sigma\leqq\sigma'$, $0\leqq t\leqq 2\pi$, and thus all the β_n with $n>n_0$ are also situated in a bounded rectangle Δ_2. There exist therefore a sequence $\{n_i\}$, a point ρ belonging to Δ, and a point β belonging to Δ_2, such that $\lim_{i=\infty}\rho_{\omega_{n_i}}=\rho$, $\lim_{i=\infty}\beta_{n_i}=\beta$, and $\gamma=\rho+\beta$. But obviously, $\rho\in S_f^{\sigma_1}$ and $\beta\in S_\varphi^{\sigma_2}$, which proves our assertion.

It is also easy to see by Picard's theorem that if $\alpha_1\in S_f^{\sigma_1}$, and if α_1 (with $\mathcal{R}(\alpha_1)>\sigma_1$) is an isolated essential singularity of $f(s)$, then, for each β belonging to $S_\varphi^{\sigma_2}$: $\alpha_1+\beta\in P[f,\sigma_1;\varphi,\sigma_2]$. At any rate, if $S_{f\varphi}$ denotes the sum-composite of $S_f^{\sigma_1}$ and $S_\varphi^{\sigma_2}$, $f(s)$ and $\varphi(s)$ being represented by Taylor-D series, then the closure of the union of the two sets $S_{f\varphi}\cup P[f,\sigma_1;\varphi,\sigma_2]$ is equal to the set $S_{f\varphi}$. But, for general Dirichlet series, it cannot even be asserted that this closure is equal to the closure of $S_{f\varphi}$. In order to have composition-theorems bearing on general Dirichlet series, without being obliged to take

into account the order e. f. s. of the functions, the introduction of sets of the type $P[f, \sigma_1; \varphi, \sigma_2]$ seems natural and indispensable. If $\sigma_1=\sigma_2=-\infty$, the set $P[f, \sigma_1; \varphi, \sigma_2] = P[f, -\infty; \varphi, -\infty]$ will be denoted simply by $P[f; \varphi]$. Let us remark that $P[f; \varphi]$ is not equal generally to $P[\varphi; f]$.

THEOREM XXXVI. [7] *Let $F(s)$ be represented by $\sum a_n e^{-\lambda_n s}(\lambda_1>1)$. Suppose that $\varphi(s)=\sum e^{-\lambda_n s}$ is uniform in the whole plane, and that $\{a_n\}$ admits a generatrix-sequence $\{d_n\}$ of an integral order k with respect to $\{\lambda_n\}$. If*

$$\limsup_{n=\infty} \frac{\log|d_n|}{n} < \infty,$$

and if $f(s)=\sum d_n e^{-ns}$ is uniform, then $F(s)$ is uniform and the only possible singularities of $F(s)$ are the points of the sum-composite of S_f and S_φ, the points of $P[\varphi; f]$, the limit-points of all these points, and the points of S_φ.

We shall first suppose that $\varphi(s)$ takes all the possible values except at most one. As for the proof of Theorem XXXIII we can show, on writing formulas analogous to (107) and (108), that $\sigma_A^F \leqq \sigma_A^f + \sigma_A^\varphi$, $\left(\sigma_A^f = \limsup_{n=\infty} \frac{\log|d_n|}{n}\right)$. We have to prove that $F(s)$ is uniform and holomorphic in the part Δ of the complement (with respect to the whole plane) of the set $\overline{S_{f\varphi} \cup S_\varphi \cup P[\varphi; f]}$ which constitutes a region containing a half-plane $\sigma>\sigma_1'$. Here $S_{f\varphi}$ is the sum-composite of S_f and S_φ. Let L be a closed simple arc of Jordan belonging to Δ having one of its extremities z^* in the half-plane $\sigma>\sigma_A^f+\sigma_A^\varphi$, and let z_0 be a point of L. There exists a positive quantity, $\delta>0$, and two distinct quantities a and b, $a=a(z_0)$, $b=b(z_0)$ such that for each $\beta \in S_\varphi$: 1) $|z_0-\beta|>\delta$, and for each $\rho_a \in E_\varphi(a,-\infty)$, each $\rho_b \in E_\varphi(b,-\infty)$, and each $\alpha \in S_f$: 2) $|z_0-\alpha-\rho_a|>\delta$, 3) $|z_0-\alpha-\rho_b|>\delta$, 4) $|z_0-\beta-\alpha|>\delta$.

It is obvious that there exists a quantity $\delta_1>0$ such that

1) and 4) hold with α, β having their precise meaning and with $\delta = \delta_1$, since if this were not true, z_0 would belong either to S_φ or to $\overline{S}_{f,\varphi}$. If, on the other hand, there were not two quantities a and b and a $\delta_2 > 0$ such that 2) and 3) hold with $\delta = \delta_2$, then for each $\delta > 0$ the inequality $|z-\beta-\rho_c| \leqq \delta$ would hold for every c, with a certain $\rho_c \in E'_\varphi(c, -\infty)$, with a certain β, except for at most one value $c = c(\delta)$. But obviously, for δ sufficiently small, $c(\delta)$ (if it does exist) will have the same value, and z_0 would belong to $\overline{P[\varphi; f]}$. The smaller of the two quantities δ_1 and δ_2 is the desired quantity δ. It follows then from the established inequalities that to every $z_0 \in L$ there corresponds an $\epsilon > 0$, $\epsilon = \epsilon(z_0)$, such that, if $z \in \overline{C(z_0, \epsilon)}$, then

$$|z-\beta| > \epsilon,\ |z-\alpha-\rho_a| > \epsilon,\ |z-\alpha-\rho_b| > \epsilon,\ |z-\beta-\alpha| > \epsilon.$$

The quantity $\epsilon = \epsilon(z_0)$ depends on z_0, but, if we denote by $\epsilon'(z_0)$ the least upper bound of all the quantities $\epsilon(z_0)$ (for $z_0 \in L$ fixed), having the properties defined above, we see, by classical reasoning (L is bounded, since it is a closed arc of Jordan), that $\underset{z_0 \in L}{\text{g. l. b.}}\ \epsilon'(z_0) = \eta > 0$. In other words, there exists a constant $\eta > 0$, independent of z_0, such that for $a = a(z_0)$, $b = b(z_0)$, every $\rho_a \in E_\varphi(a(z_0), -\infty)$, $\rho_b \in E_\varphi(b(z_0), -\infty)$, $\alpha \in S_f$, $\beta \in S_\varphi$:

(112) $|z-\beta| > \eta,\ |z-\alpha-\rho_a| > \eta,\ |z-\alpha-\rho_b| > \eta, |z-\beta-\alpha| > \eta,$ when $z \in \overline{C(z_0, \eta)}$, with $z_0 \in L$.

Let us choose a finite number of points $z_0^{(1)}, z_0^{(2)}, \cdots z_0^{(k)}$ such that $z_0^{(m+1)} \in C\left(z_0^{(m)}, \frac{\eta}{2}\right)$, $(1 \leqq m \leqq k-1)$, $z_0^{(j)} \in L$, $(1 \leqq j \leqq k)$, $z_0^{(1)} = z^*$, the other extremity z_1^* of L being such that $z_1^* \in \overline{C\left(z_0^{(k)}, \frac{\eta}{2}\right)}$.

Obviously $\varphi(s) \to 0$, uniformly with respect to t as $\sigma \to \infty$ (since $\lambda_1 > 0$, here $\lambda_1 > 1$). But it is easily seen (this is a

well known fact for each Dirichlet series with $\lambda_1 > 0$) that for σ sufficiently large $\varphi(s) \neq 0$. Indeed, for σ large,

$$\varphi(s) = \sum_1^\infty e^{-\lambda_n s} = e^{-\lambda_1 s}(1 + \sum_2^\infty e^{-(\lambda_n - \lambda_1)s}),$$

and, since $\sum_2^\infty e^{-(\lambda_n - \lambda_1)s} \to 0$, as $\sigma \to \infty$, uniformly with respect to t, we see that $\varphi(s) \neq 0$, for σ large. Therefore there exists a σ_0 such that $\mathcal{R}(\rho_{a(z_0^{(j)})}) < \sigma_0$, $\mathcal{R}(\rho_{b(z_0^{(j)})}) < \sigma_0$, and if $\sigma_1 + \epsilon$ is negative, numerically sufficiently large, and if $s = \sigma_1 + \epsilon + it$, then, for each $z \in \overline{C(z_0^{(j)}, \eta)}$, $(1 \leqq j \leqq k)$, each $\beta \in S_\varphi$, and each $a = a(z_0^{(j)})$, $b = b(z_0^{(j)})$, $(1 \leqq j \leqq k)$:

(113) $\quad |z - s - \rho_a| > \eta, \quad |z - s - \rho_b| > \eta, \quad |z - s - \beta| > \eta.$

Let now $C_1^*(\epsilon)$ have the same meaning as in formula (110).[1] From what we have seen (inequalities (112) and (113)), if ϵ is sufficiently small, the inequalities (113) hold if s is any point of $C_1^*(\epsilon)$ (σ_1 is chosen negative, numerically sufficiently large), moreover (still if ϵ is sufficiently small), we shall have $\mathcal{R}(z^* - s) \geqq \mathcal{R}(z^*) - \sigma_A{}^f - 2\epsilon > \sigma_A{}^\varphi + \epsilon$. If s is a fixed point of $C_1^*(\epsilon)$, the point $u = z - s$, when z varies on L, varies on a curve $L(s)$, which is obtained from L by translation. One extremity $u^* = z^* - s$, of $L(s)$ is in the half-plane $\sigma > \sigma_A{}^\varphi + \epsilon$; we have thus $|\varphi(u^*)| = |\varphi(z^* - s)| \leqq M = \sum e^{-\lambda_n(\sigma_A{}^\varphi + \epsilon)}$, for each $s \in C_1^*(\epsilon)$.

By Schottky's theorem, if $\Phi(z)$ is holomorphic in a circle $\overline{C(A, R)}$, if $|\Phi(A)| < N$, if $\Phi(z)$ does not take in $\overline{C(A, R)}$ two distinct values a and b, and if $0 < \theta < 1$, then

$$|\Phi(z)| < K(a, b;\ N;\ \theta) < \infty$$

in $\overline{C(A, \theta R)}$. Therefore, if for s fixed we set $u_2^* = z_0^{(2)} - s$, $\cdots$, $u_k^* = z_0^{(k)} - s$, we shall have:

$$|\varphi(u)| < K(a(z_0^{(1)}),\ b(z_0^{(1)});\ M;\ \tfrac{1}{2}) = K_1,$$

for $u \in \overline{C\left(u^*, \frac{\eta}{2}\right)}$, that is to say, also: $|\varphi(u_2^*)| < K_1$. On ap-

[1] We define here $C_1^*(\epsilon)$ with respect to the function $f(s)$ of the actual statement as we did for the function $f(s)$ of Theorem XXXIII.

plying once more Schottky's theorem we shall have

$$|\varphi(u)| < K(a(z_0^{(2)}),\ b(z_0^{(2)});\ K_1;\ \tfrac{1}{2}) = K_2,$$

for $u \in C\left(u_2^*, \frac{\eta}{2}\right)$, that is to say, also $|\varphi(u_3^*)| < K_3$, etc. . . .

In applying Schottky's theorem k times, we see that $|\varphi(z-s)| < B$, when z is in a channel of which L is the central line, s being an arbitrary point of $C_1^*(\epsilon)$, B being independent of z and s.

It is now easy to see that the integral in (110), once more, defines a function holomorphic on L. The remaining part of the proof is obvious.

We have supposed that $\varphi(s)$ takes all possible values except at most one. If $\varphi(s)$ does not take two distinct values then, as the reader can see easily, the proof becomes simpler, since then, if a_1 and b_1 are the two values which $\varphi(s)$ does not take, the last passage of the proof, where Schottky's theorem is used, can be applied directly with $a(z_0^{(j)}) = a_1$, $b(z_0^{(j)}) = b_1$, $(1 \leqq j \leqq k)$.

A more general theorem of the same kind can be proved, in exactly the same manner, if we consider series of the form $\sum d_n e^{-ns}$, $\sum b_n e^{-\lambda_n s}$, and $\sum a_n e^{-\lambda_n s}$ where $a_n = b_n \sum_{m<\lambda_n} (\lambda_n - m)^k d_n$. H. Brunk, in his doctoral thesis [**5**] proved theorems of a very general type in which points of a character similar to those of a set $P[f, \varphi]$ are involved.

Theorems on singularities of Taylor series can also be applied in a reverse sense, so to speak, in order to assure that a function represented by a Dirichlet series of a general type necessarily admits singularities of given affixes.

Let $f(s)$ be represented by $\sum a_n e^{-\lambda_n s}$, and let k be a positive integer. The n^{th} coefficient of $f(s)$ with respect to the sequence $\{n\}$, of order k, shall be called the n^{th} *Taylor-*

coefficient of $f(s)$, *of order* k. This quantity shall be denoted by $a_T^{(k)}(n)$.

THEOREM XXXVII. *Suppose that* $f(s)=\sum a_n e^{-\lambda_n s}$, *with* $\sigma_A{}^f<\infty$, *is uniform in a half-plane* $\sigma>\sigma_1$, *and that* $f(s)$ *is of order* ν *e. f. s. in this half-plane. Let* β *be a singularity of the function* $F(s)$ *represented by* $\sum a_T^{(k)}(n)e^{-ns}$, *where* $k>\nu$, *and let us suppose that* β *is not of the form* $2m\pi i$, *where* m *is an integer. To each* $\epsilon>0$ *there corresponds an integer* p *such that there exists a point* α *belonging to* $S_f{}^{\sigma_1}$ *with the property*:

$$|\alpha-(2p\pi i+\beta)|<\epsilon.$$

Indeed, the function $\varphi(s)=\sum e^{-ns}$ is uniform in each half-plane $\sigma>\sigma_2$, and it is bounded e. f. s. in such a half-plane. Therefore Theorem XXXIII can be applied to $f(s)$ and $\varphi(s)$, with $k>\nu$. Hence the only possible singularities of $F(s)$, in the whole plane, are the points of the sum-composite of $S_f{}^{\sigma_1}$ and S_φ, the limit-points of these points, and the points of S_φ. But S_φ is composed of the points of affixes $2m\pi i$, where m takes all integral values. If β is a singularity of $F(s)$, and is not of the form $2m\pi i$, that is to say, does not belong to S_φ, it must belong either to the sum-composite of $S_f{}^{\sigma_1}$ and S_φ or has to be a limit-point of this sum-composite. In other words, either there exist a point α_1 belonging to $S_f{}^{\sigma_1}$ and an integer m_1 such that $\beta=\alpha_1+2m_1\pi i$; or there exist a sequence $\{\alpha_n\}$, with $\alpha_n \in S_f{}^{\sigma_1}$, $(n\geqq 1)$, and a sequence of integers $\{m_n\}$ such that $\beta=\lim_{n=\infty}(\alpha_n+2m_n\pi i)$. In the second case, if $\epsilon>0$ is given, for n sufficiently large the inequality $|\beta-\alpha_n-2m_n\pi i|<\epsilon$ is satisfied. In both cases the statement of the theorem is satisfied.

Theorem XXXVII allows us to associate with each theorem on Taylor series furnishing a condition in order that a given point be singular for the corresponding function, a new theorem on Dirichlet series of general type (of finite

order e. f. s. in a half-plane) locating some of its singularities.

This and other theorems of this type were proved by the author [**8**].

Since these lectures were delivered, H. Brunk has proved, in his work mentioned above [**5**], many interesting and important theorems related with the subject of this chapter. He gives in fact a generalization of Theorem XXXII, in which he replaces the boundedness e. f. s. in a half-plane by a less restrictive condition, but conserving the form $\sum a_n b_n e^{-\lambda_n s}$ of the composite function.

He supposes, for instance, that in the part $\Delta(\sigma_2)$ of the half-plane $\sigma > \sigma_2$ from which the points $S_\varphi^{\sigma_2}$ are excluded $\varphi(s)$ has the following property: to each positive δ there correspond two positive quantities $L(\delta)$ and $M(\delta)$ such that if $s_1 \in \Delta(\sigma_2)$, and if there exists a channel connecting s_1 to the half-plane $\sigma > \sigma_A^\varphi$, of width 2δ, which lies in $\Delta(\delta_2)$, and of which the central line is not larger than $L(\delta)$, then $|\varphi(s_1)| < M(\delta)$. A similar property is supposed to be satisfied by $f(s)$ in a quarter-plane: $\sigma > \sigma_1$, $t > t_0$. The conclusion of Brunk's theorem is similar to that of Theorem XXXII. He made also an extensive study of cases in which the sets $S_f^{\sigma_1, t_0}$ and $S_\varphi^{\sigma_2}$ can be replaced by larger sets, by adding, for instance, to $S_\varphi^{\sigma_2}$ the set in which $\varphi(s)$ takes two distinct values, in order to obtain composition-theorems in which neither the boundedness nor other similar conditions are any longer necessary. Such theorems generalize Theorem XXXVI.

Brunk gave also many interesting applications of Theorem XXXII, as well as of its generalizations. We shall state here a particular case of one of his theorems [**5**].

THEOREM XXXVIII. *If $f(s) = \sum a_n e^{-\lambda_n s}$ is uniform and bounded e. f. s. for $\sigma > \sigma_1$, with $\sigma_1 < \sigma_A^f (|\sigma_A^f| < \infty)$, if the only singularities of $f(s)$ in the half-plane $\sigma > \sigma_1$ are poles of affixes*

$s_0+2k\pi i$, where $s_0=\sigma_A{}^f+it_0$, if there exists a quantity M such that the orders of these poles are all smaller than M, if $\lim_{n=\infty}\inf(\lambda_{n+1}-\lambda_n)>0$, $\lim_{n=\infty}\dfrac{\log|a_n|}{\lambda_n}=\sigma_A{}^f$, *then there exists an integer $m>0$, such that*

$$f(s)=\sum_{q=0}^{m} e^{-\lambda' qs}T_q(s),$$

where $T_q(s)$ is a Taylor-D series.

For the proof of this theorem and for many other interesting results we refer the reader to Brunk's Thesis [**5**].

X

COMPOSITE THEOREMS OF HURWITZ TYPE

A composition-theorem on Taylor series was also given by Hurwitz in 1899 (see, for instance, [**6**]). We shall state his theorem as elliptically as we did for Hadamard's theorem at the beginning of Chapter IX.

If E_1, E_2, E_3 are, respectively, the sets of singularities of the functions represented by the series $\sum_0^\infty \frac{a_n}{z^{n+1}}$, $\sum_0^\infty \frac{b_n}{z^{n+1}}$, $\sum_0^\infty \frac{c_n}{z^{n+1}}$, *where* $c_n = a_n b_0 + C_n^1 a_{n-1} b_1 + \cdots + a_0 b_n$, *and if* $c \in E_3$, *then there exists an* a *belonging to* E_1 *and a* b *belonging to* E_2 *such that* $c = a + b$.

If $z = 0$ is a regular point for the function $\psi(z)$ represented by $\sum_0^\infty \frac{a_n}{z^{n+1}}$, we shall say that the function $f(s)$ represented by $\sum a_n e^{-(n+1)s}$ is regular at $-\infty$. We shall say that $-\infty$ is a singularity for $f(s)$, if $z = 0$ is a singularity for $\psi(z)$.

On setting $z = e^s$, the theorem of Hurwitz becomes:

If $f(s) = \sum_0^\infty a_n e^{-(n+1)s}$, $\varphi(s) = \sum_0^\infty b_n e^{-(n+1)s}$ and $F(s) = \sum_0^\infty c_n e^{-(n+1)s}$, with $c_n = a_n b_0 + C_n^1 a_{n-1} b_1 + \cdots + a_0 b_n$, then $F(s)$ has no other singularities than the points γ of the form $\gamma = \log(e^\alpha + e^\beta)$, where α is a point of S_f and where β is a point of S_φ. Of course, if $\alpha = -\infty$ is a singularity for $f(s)$, this point should be considered as belonging to S_f, and each point $\log(e^{-\infty} + e^\beta) = \beta$ belonging to S_φ should be considered as a possible singularity for $F(s)$. In the same way, if $\beta = -\infty$ is singular for $\varphi(s)$, each point α belonging to S_f is a possible singularity for $F(s)$.

Let now $\theta(s) \equiv \sum d_n e^{-\lambda_n s}$ be a Dirichlet series representing a uniform function. If the following conditions are satisfied: 1) there exists a quantity σ_0 such that $\theta(s)$ is holomorphic in each finite region contained in the half-plane $\sigma < \sigma_0$, 2) $\theta(s)$ tends to a finite limit, which we shall denote by $\theta(-\infty)$, as $\sigma \to -\infty$, uniformly with respect to $t(-\infty < t < \infty)$, 3) $(\theta(s) - \theta(-\infty))e^{-s}$ tends to a finite limit, which we shall denote by $\theta'(-\infty)$, as $\sigma \to -\infty$, uniformly with respect to t—we shall say that $s = -\infty$ is a regular point for $\theta(s)$. If one of these conditions is not satisfied we shall say that $s = -\infty$ is a singularity for $\theta(s)$, or that $-\infty \in S_\theta$.

A simple generalization of Hurwitz' theorem can be obtained, if one of the two series is a Taylor-*D* series, the other one being of an arbitrary type.

THEOREM XXXIX. *Suppose that* $f(s) = \sum_0^\infty a_n e^{-(n+1)s}$, *with* $\sigma_A^f < \infty$, *and* $\varphi(s) = \sum_0^\infty b_n e^{-\lambda_{n+1} s}$, *with* $\sigma_A^\varphi < \infty$, *are both uniform in the whole plane. Let* $\{l_n\}$, $(n \geqq 1)$ *be the sequence of all the sums* $\lambda_r + m$, $(r \geqq 1,\ m \geqq 0)$, *written in an increasing order, and let us set*

$$d_n = \sum\sum a_m b_r A(r, m),$$

where for $m \geqq 1$:

$$A(r, m) = \frac{\lambda_{r+1}(\lambda_{r+1}+1)\cdots(\lambda_{r+1}+m-1)}{m!};\ (r \geqq 1,\ m \geqq 1),$$

and where $A(r, 0) = 1$, *the double series being taken with respect to all* $r \geqq 0$, *and all* $m \geqq 0$, *such that* $\lambda_{r+1} + m = l_{n+1}$. *The series* $\sum_0^\infty d_n e^{-l_{n+1} s}$ *has an abscissa of absolute convergence which is not larger than* $\log(e^{\sigma_A^f} + e^{\sigma_A^\varphi})$, *the function* $F(s) = \sum_0^\infty d_n e^{-l_{n+1} s}$ *is uniform, and the only possible singularities of* $F(s)$ *are the points* γ *such that* $\gamma = \log(e^\alpha + e^\beta)$, *where* α *is any point of* S_f

and β any point of S_φ and the limit-points of this set.[1] *In* $\log(e^\alpha+e^\beta)$ *each determination of the logarithm should be taken.*

Let $c>\sigma_A^f$, and let us prove that, for x sufficiently large:

$$F(z)=\int_c^{c+2\pi i} f(s)e^s\varphi[\log(e^z-e^s)]\,ds, \tag{114}$$

where (for x sufficiently large, $\sigma=c$)

$$\varphi[\log(e^z-e^s)]=\sum_0^\infty b_n(e^z-e^s)^{-\lambda_{n+1}}=\sum_{n=0}^\infty b_n e^{-\lambda_{n+1}z}(1-e^{(s-z)})^{-\lambda_{n+1}}$$

$$=\sum_{n=0}^\infty b_n e^{-\lambda_{n+1}z}\left[1+\sum_{m=1}^\infty \frac{\lambda_{n+1}(\lambda_{n+1}+1)\cdots(\lambda_{n+1}+m-1)}{m!}e^{m(s-z)}\right],$$

the path of integration being the straight line segment joining the points c to $c+2\pi i$. If $\log(e^z-e^c)>a>\sigma_A^\varphi$:

$$\frac{1}{2\pi i}\int_c^{c+2\pi i} f(s)\varphi[\log(e^z-e^s)]e^s\,ds=$$

$$\frac{1}{2\pi i}\int_c^{c+2\pi i} f(s)\sum_{n=0}^\infty b_n e^{-\lambda_{n+1}z}\left[\sum_{m=1}^\infty \frac{\lambda_{n+1}(\lambda_{n+1}+1)\cdots(\lambda_{n+1}+m-1)}{m!}e^{m(s-z)}+1\right]e^s\,ds$$

$$=\sum_{n=0}^\infty\sum_{m=1}^\infty b_n\frac{\lambda_{n+1}(\lambda_{n+1}+1)\cdots(\lambda_n+m-1)}{m!}e^{-(\lambda_{n+1}+m)z}\cdot\frac{1}{2\pi i}\int_c^{c+2\pi i} f(s)e^{(m+1)s}\,ds$$

$$+\sum_{n=0}^\infty b_n e^{-\lambda_{n+1}z}\cdot\frac{1}{2\pi i}\int_c^{c+2\pi i} f(s)e^s\,ds$$

$$=\sum_{n=0}^\infty\sum_{m=1}^\infty b_n a_m\frac{\lambda_{n+1}(\lambda_{n+1}+1)\cdots(\lambda_{n+1}+m-1)}{m!}e^{-(\lambda_{n+1}+m)z}$$

$$+a_0\sum_{n=0}^\infty b_n e^{-\lambda_{n+1}z}=\sum_{n=0}^\infty d_n e^{-l_{n+1}z}.$$

Since these equalities still hold if $\sum a_n e^{-(n+1)s}$ and $\sum b_n e^{-\lambda_{n+1}s}$ are respectively replaced by $\sum|a_n|e^{-(n+1)s}$ and $\sum|b_n|e^{-\lambda_{n+1}s}$, we see that $\sum d_n e^{-l_{n+1}s}$ converges absolutely for $x>\log(e^c+e^{\sigma_A^\varphi})$, and since c is chosen arbitrarily such that $c>\sigma_A^f$, we see that $\sigma_A^F\leqq\log(e^{\sigma_A^f}+e^{\sigma_A^\varphi})$.

[1] If $f(s)$ admits $-\infty$ as a singularity, then each point $\gamma=\log(e^{-\infty}+e^\beta)=\beta+2k\pi i$, with $\beta\in S_\varphi$, should be considered as a possible singularity of $F(s)$. If $-\infty$ is a singularity for $\varphi(s)$, each point $\alpha+2k\pi i$, $\alpha\in S_f$, is a possible singularity of $F(s)$.

Let σ_1 be an arbitrary quantity such that $\sigma_1 < \sigma_A^f$ and let $(S_f^{\sigma_1})$ be the part of $S_f^{\sigma_1}$ for the points of which $0 \leqq t \leqq 2\pi$. For $\epsilon > 0$, let $(S_f^{\sigma_1}(\epsilon))$ be the union of a finite number of closed circles of radius ϵ containing inside the points of $(S_f^{\sigma_1})$.[1] Let $D(\epsilon, \sigma_1)$ be the region composed of the points of the rectangle $\sigma_1 \leqq \sigma \leqq c$, $0 \leqq t \leqq 2\pi$ which are not points of $(S_f^{\sigma_1}(\epsilon))$ and for which the points $\sigma = c$, $0 \leqq t \leqq 2\pi$ are boundary points (ϵ being sufficiently small). If z, with x sufficiently large, is fixed, the function of s: $\Phi_z(s) = f(s)\varphi[\log(e^z - e^s)]e^s$ is holomorphic in $D(\epsilon, \sigma_1)$, the function $\varphi[\log(e^z - e^s)]$ being the analytic continuation, on paths which (s varying only) lie inside or on the boundary of $D(\epsilon, \sigma_1)$, of the function given, for s on $\sigma = c$, by the double series written above. Moreover, the function $\Phi_z(s)$ takes the same value for two values σ' and $\sigma' + 2\pi i$ which belong to the boundary $C(\epsilon, \sigma_1)$ of $D(\epsilon, \sigma_1)$.

Therefore, if we denote by C_{ϵ, σ_1} the part of $C(\epsilon, \sigma_1)$ composed of the points which are neither points of $\sigma = c$, nor points of the lines $t = 0$ or $t = 2\pi$, we have by Cauchy's theorem:

$$F(z) = \frac{1}{2\pi i}\int_{C_{\epsilon, \sigma_1}} f(s)\varphi[\log(e^z - e^s)]e^s ds. \tag{115}$$

Now let z vary in a bounded closed region Δ containing points with abscissa x such that $x > \log(e^c + e^{\sigma_A^\varphi})$, and such that for every $z \in \Delta$, every $s \in C_{\epsilon, \sigma_1}$ and every $\beta \in S_\varphi$, $\beta \neq -\infty$,[2] the inequalities

$$\begin{aligned} &|z - \log(e^s + e^\beta)| > \epsilon_1 \\ &|z - s + 2k\pi i| > \epsilon_2 \ (k \text{ any integer}) \end{aligned} \tag{116}$$

[1] $(S_f^{\sigma_1}(\epsilon))$ can be obtained by taking the union of all the circles of radius ϵ around each point of $(S_f^{\sigma_1})$ and extracting, by Borel-Lebesgue's theorem, a finite number of these circles.

[2] If $-\infty \in S_\varphi$.

are satisfied, the positive quantities ϵ_1, ϵ_2 being arbitrarily fixed, the logarithm being taken with all its possible determinations. Then obviously for a certain $\delta>0$:

$$|e^z-e^s-e^\beta|>\delta,$$

and

$$|\log(e^z-e^s)-\beta|>\epsilon'_1=\epsilon'_1(\epsilon_1),$$
$$\mathcal{R}\log(e^z-e^s)=\log|e^z-e^s|>N=N(\epsilon_2),$$

where $\epsilon'_1>0$, $N>-\infty$. The function $\varphi[\log(e^z-e^s)]$, where $z\in\Delta$, $s\in C_{\epsilon,\sigma_1}$ is then a holomorphic function of z and s, and $F(z)$, given by (115), is a holomorphic function of z in Δ. If, moreover, $s=-\infty$ is a regular point for $\varphi(s)$ then $F(z)$ is holomorphic in each bounded closed region Δ containing points with x such that $x>\log(e^c+e^{\sigma_A(\varphi)})$ and such that for every $z\in\Delta$, every $s\in C_{\epsilon,\sigma_1}$ and every $\beta\in S_\varphi$, the first inequality of (116) is satisfied. Indeed, the function of z, $\varphi[\log(e^z-e^s)]$ is then holomorphic in Δ, if we give to this function the value $\varphi(-\infty)$ when $z=s$, $z\in\Delta$, since this function is then continuous in z (for s fixed), and moreover it has a derivative with respect to z at each point $z\in\Delta$. This is obvious for $z=z_0\in\Delta$, if $e^{z_0}\neq e^s$, but, if $e^{z_0}=e^s$ then:

$$\left.\frac{d\varphi[\log(e^z-e^s)]}{dz}\right|_{z=z_0}=\lim_{z=z_0}\frac{\varphi[\log(e^z-e^{z_0})]-\varphi(-\infty)}{z-z_0}$$
$$=\lim_{z=z_0}\frac{\varphi[\log(e^z-e^{z_0})]-\varphi(-\infty)}{e^z-e^{z_0}}\lim_{z=z_0}\frac{e^z-e^{z_0}}{z-z_0}=\varphi'(-\infty)e^{z_0}.$$

From the fact that $\varphi[\log(e^z-e^s)]$ is holomorphic in Δ, for each fixed value of s belonging to C_{ϵ,σ_1}, it follows then, by a well known theorem, that $F(z)$ given by (115) is holomorphic in Δ.

It should be emphasized that the set $(S_f^{\sigma_1})$ (which is the part of $S_f^{\sigma_1}$ contained in the strip $0\leqq t\leqq 2\pi$) contains the points of the segment L_{σ_1} given by $\sigma=\sigma_1$, $0\leqq t\leqq 2\pi$. Let $\sum_{\sigma_1}(\epsilon)$ be the set of the points of the circles which form a part of $(S_f^{\sigma_1}(\epsilon))$ and which are such that each point of L_{σ_1}

is inside of one of these circles. If $f(s)$ is regular at $s=-\infty$, for $-\sigma_1$ sufficiently large, $\sum_{\sigma_1}(\epsilon)$ is separated from the set $S_f^{*\sigma_1}(\epsilon)$, formed by the circles, which form a part of $(S_f^{\sigma_1}(\epsilon))$, and which are such that each singularity of $f(s)$ contained in the strip $0\leqq t\leqq 2\pi$ is contained in one of these circles (ϵ being fixed). The set C_{ϵ,σ_1} is composed, for $-\sigma_1$ sufficiently large, of an arc l_{ϵ,σ_1} composed of arcs of circles of radius ϵ and centers at $L_{\sigma_1}(\epsilon)$, and the set $C_{\epsilon,\sigma_1}-l_{\epsilon,\sigma_1}$. The affixes of l_{ϵ,σ_1} (always provided that $s=-\infty$ is regular for $f(s)$) tend to infinity as $\sigma_1\to-\infty$ in such a way that if $s'=\sigma'+it'\ \epsilon\ l_{\epsilon,\sigma_1}$, $\sigma'\to-\infty$ uniformly with respect to t'. The set $C'_{\epsilon,\sigma_1}=C_{\epsilon,\sigma_1}-l_{\epsilon,\sigma_1}$ does not vary if $-\sigma_1$ is sufficiently large. On choosing z, as above, so as to have $x>\log(e^c+e^{\sigma_A^{\varphi}})$, we may write (115), for $-\sigma_1$ sufficiently large:

$$F(z)=\frac{1}{2\pi i}\int_{l_{\epsilon,\sigma_1}} f(s)e^{s}\varphi[\log(e^{z}-e^{s})]ds$$

(117)

$$+\frac{1}{2\pi i}\int_{C'_{\epsilon,\sigma_1}} f(s)e^{s}\varphi[\log(e^{z}-e^{s})]ds.$$

But since, if $f(s)$ is regular at $s=-\infty$, $f(s)e^{s}\to 0$ as $\sigma\to-\infty$, uniformly with respect to $0\leqq t\leqq 2\pi$, and since the second integral is independent of σ_1, if $-\sigma_1$ is sufficiently large, we see that, if $f(s)$ is regular at $s=-\infty$, then for $-\sigma_1$ sufficiently large ($\epsilon>0$ being fixed):

$$F(z)=\frac{1}{2\pi i}\int_{C'_{\epsilon,\sigma_1}} f(s)e^{s}\varphi[\log(e^{z}-e^{s})]ds.$$

This proves that, if $f(s)$ is regular at $s=-\infty$, then: a) if z varies in a bounded region Δ containing points with $x>\log(e^c+e^{\sigma_A^{\varphi}})$, and such that for every $s\ \epsilon\ C'_{\epsilon,\sigma_1}$ and every $\beta\ \epsilon\ S_{\varphi}$, $\beta\neq-\infty$ the inequalities (116) hold, or: b) if $s=-\infty$ is regular for $\varphi(s)$ and if we suppose only that the first of the inequalities (116) holds—then $F(z)$ is holomorphic.

This remark, added to the remarks made above when $F(z)$ was defined by (115), completes, on taking into account that $\epsilon>0$, $\epsilon_1>0$ and $\epsilon_2>0$ are arbitrary, the proof of the theorem.

The reader will see for himself that Theorem XXXIX contains Hurwitz' theorem as a particular case: $\lambda_n=n$ $(n\geqq 1)$.

A theorem of a less precise form was proved by the author in his paper of 1929 [**8**].

A theorem of a more general character, bearing on two Dirichlet series of general and different types: $\sum a_n e^{-\lambda_n s}$, $\sum b_n e^{-\mu_n s}$ can be obtained if the notion of finite order e. f. s. is introduced.

THEOREM XL. *Let* $f(s)=\sum_1^\infty a_n e^{-\mu_n s}$ *be uniform and of order* ν *e. f. s. in the half-plane* $\sigma>\sigma_1$, *and let* $\varphi(s)=\sum_1^\infty b_n e^{-\lambda_n s}$, *with* $\sigma_A^\varphi<\infty$, *be uniform and of order* μ *e. f. s. in the half-plane* $\sigma>\sigma_2$. *Let* k *be an integer larger than* $\nu+\mu$, *let* $a_T^{(k)}(n)$ *be the* n^{th} *Taylor coefficient of order* k, *let* $\{l_n\}$, $(n\geqq 1)$, *be the sequence of all the sums* λ_r+m, $(r\geqq 1,\ m\geqq 1)$, *written in an increasing order, and let*

$$d_n=\sum\sum a_T^{(k)}(m)b_r\frac{\lambda_r(\lambda_r+1)\cdots(\lambda_r+m-1)}{m!},$$

the double series being taken with respect to all $r\geqq 1$, $m\geqq 1$, *such that* $\lambda_r+m=l_n$. *The series* $\sum d_n e^{-l_n s}$ *has an abscissa of absolute convergence not larger than* $\log(e^{\mathrm{Max}(\sigma_A f,\,0)}+e^{\sigma_A^\varphi})$, *the function* $F_k^*(s)=\sum d_n e^{-l_n s}$ *is uniform in the half-plane* $\sigma>\log(e^{\mathrm{Max}(H_f(\sigma_1),\,0)}+e^{\sigma_2})$, *and the only possible singularities of* $F_k^*(s)$ *in this half-plane are the points of the form* $\gamma=\log(e^\alpha+e^\beta)$, *where* α *is a point of* $S_f^{\sigma_1}$, β *a point of* $S_\varphi^{\sigma_2}$, *the limit-points of these points* γ, *the points of the form* $\gamma=\log(e^\beta+1)$ *and the limit-points of this set. The logarithms*

in both expressions $\log(e^{\alpha}+e^{\beta})$, $\log(e^{\beta}+1)$ *should be taken with all their determinations.*

Let $c>\max(0, \sigma_A^f)$, we shall prove that, for x sufficiently large,

$$F_k^*(z)=\frac{k!}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\varphi[\log(e^z-e^s)]\frac{ds}{s^{k+1}}, \tag{118}$$

where (for x sufficiently large) $\varphi[\log(e^z-e^s)]=\sum_1^{\infty} b_n(e^z-e^s)^{-\lambda_n}$

$$=\sum_{n=1}^{\infty} b_n e^{-\lambda_n z}\Big[1+\sum_{m=1}^{\infty}\frac{\lambda_n(\lambda_n+1)\cdots(\lambda_n+m-1)}{m!}e^{m(s-z)}\Big],$$

the path of integration being the line $\sigma=c$.

We have indeed, by Theorem X, for x sufficiently large:

$$\frac{k!}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\varphi[\log(e^z-e^s)]\frac{ds}{s^{k+1}}$$

$$=\frac{k!}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\sum_{n=1}^{\infty} b_n e^{-\lambda_n z}\Big[1+\sum_{m=1}^{\infty}\frac{\lambda_n(\lambda_n+1)\cdots(\lambda_n+m-1)}{m!}e^{m(s-z)}\Big]\frac{ds}{s^{k+1}}$$

$$=\sum_{n=1}^{\infty} b_n\sum_{m=1}^{\infty}\frac{\lambda_n(\lambda_n+1)\cdots(\lambda_n+m-1)}{m!}e^{-(\lambda_n+m)z}\cdot\frac{k!}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)e^{ms}\frac{ds}{s^{k+1}}$$

$$+\sum_{n=1}^{\infty} b_n e^{-\lambda_n z}\frac{k!}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s)\frac{ds}{s^{k+1}}$$

$$=\sum_{n=1}^{\infty} b_n\sum_{m>\mu_1}^{\infty}\frac{\lambda_n(\lambda_n+1)\cdots(\lambda_n+m-1)}{m!}e^{-(\lambda_n+m)z}\cdot\sum_{\mu_n\leq m}(m-\mu_n)^k a_n$$

$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_T^{(k)}(m)b_n\frac{\lambda_n(\lambda_n+1)\cdots(\lambda_n+m-1)}{m!}e^{-(\lambda_n+m)z}$$

$$=\sum_1^{\infty} d_n e^{-l_n z}.^{1}$$

These equalities hold if $\log(e^z-e^c)>\sigma_A^{\varphi}$, that is to say, if $x>\log(e^c+e^{\sigma_A^{\varphi}})$. They hold for the same values of z if $\sum a_n e^{-\mu_n s}$ and $\sum b_n e^{-\lambda_n s}$ are respectively replaced by $\sum|a_n|e^{-\mu_n s}$ and $\sum|b_n|e^{-\lambda_n s}$. This proves that $\sum_1^{\infty} d_n e^{-l_n s}$ converges absolutely for $x>\log(e^c+e^{\sigma_A^{\varphi}})$, and since c is chosen arbitrarily,

[1] It should be recalled that, if $m\leq\mu_1$, then $a_T^{(k)}(m)=0$.

such that $c > \max(0, \sigma_A^f)$, we see that $\sigma_A^F \leqq \log(e^{\max(\sigma_A^f, 0)} + e^{\sigma_A^\varphi})$.

Let us now suppose that $c > \max(0, \sigma_1, \sigma_A^f)$. If we define $C_1^*(\epsilon)$ in exactly the same way as we did in proving Theorem XXXIII, we see, in the same way as we proved (110), that, this time for $x > \log(e^c + e^{\sigma_A^\varphi})$:

$$F_k^*(s) = \frac{k!}{2\pi i} \int_{C_1^*(\epsilon)} f(s)\varphi[\log(e^s - e^s)] \frac{ds}{s^{k+1}}. \tag{119}$$

Now let z vary in a closed bounded region Δ which contains points with $x > \log(e^c + e^{\sigma_A^\varphi})$, and such that, for $\epsilon > 0$ arbitrary, fixed, for each $z \in \Delta$, each $\alpha \in S_f^{\sigma_1}$, each $\beta \in S_\varphi^{\sigma_2}$, and each determination of the logarithm:

$$|z - \log(e^\alpha + e^\beta)| > \epsilon_1. \tag{120}$$

Let also, for each $z \in \Delta$, the following inequality be satisfied:

$$x > \log(e^{\mathrm{Max}(H_f(\sigma_1), 0)+2\epsilon} + e^{\sigma_2}). \tag{121}$$

Since for each $\alpha \in S_f^{\sigma_1}$ and each $s \in C_1^*(\epsilon)$ the inequalities $|e^\alpha| \leqq e^{\sigma_A^f}$, $|e^s| \leqq e^{\sigma_A^f + 2\epsilon}$ are satisfied and, since to each $s \in C_1^*(\epsilon)$ there corresponds an $\alpha \in S_f^{\sigma_1}$, such that $|s - \alpha| \leqq 2\sqrt{2}\epsilon$, we see that to each $s \in C_1^*(\epsilon)$ there corresponds an $\alpha \in S_f^{\sigma_1}$ such that:

$$|e^s - e^\alpha| < \eta = \eta(\epsilon), \tag{122}$$

where $\eta(\epsilon) \to 0$ as $\epsilon \to 0$.

There exists therefore a constant $\delta > 0$ such that, if ϵ is sufficiently small, the inequality

$$|\log(e^z - e^s) - \beta| > \delta \tag{123}$$

is satisfied for each $z \in \Delta$, each $s \in C_1^*(\epsilon)$, and each $\beta \in S_\varphi^{\sigma_2}$, the logarithm being taken with all its determinations.

Indeed, if (123) were not true there would exist a sequence $\{z_i\}$, $z_i \in \Delta$, a sequence $\{s_i\}$, $s_i \in C_1^*(\epsilon)$, a sequence $\{\beta_i\}$, $\beta_i \in S_\varphi^{\sigma_2}$, such that:

$$\lim_{i=\infty} (\log(e^{z_i} - e^{s_i})) = 0, \tag{124}$$

(the determination of the logarithm being suitably chosen for each i). And, since there exist three constants, M_1, M_2,

M_3 such that $|\mathcal{R}(z)| < M_1$, $z \in \Delta$, $|\mathcal{R}(s)| < M_2$, $s \in C_1^*(\epsilon)$, $|\mathcal{R}(\beta)| < M_3$, $\beta \in S_\varphi^{\sigma_2}$, the equality (122) would imply:

$$\lim_{i=\infty} (e^{z_i} - e^{s_i} - e^{\beta_i}) = 0,$$

and, by (122), for i sufficiently large, we would have for a certain sequence $\{\alpha_i\}$:

$$|e^{z_i} - e^{\alpha_i} - e^{\beta_i}| \leqq 2\eta(\epsilon),$$

which would also furnish the inequality (for ϵ sufficiently small):

$$|z_i - \log(e^{\alpha_i} + e^{\beta_i})| < \eta_1 = \eta_1(\epsilon),$$

with $\eta_1(\epsilon) \to 0$ as $\epsilon \to 0$. And for ϵ sufficiently small (ϵ_1 being fixed) this would be in contradiction with (120). Thus (123) does hold. It follows, on the other hand, from (121) that, if $z \in \Delta$, and $s \in C_1^*(\epsilon)$, then:

$$\begin{aligned} \mathcal{R}[\log(e^z - e^s)] &= \log|e^z - e^s| \geqq \log(e^z - e^{\sigma}) \\ &\geqq \log(e^z - e^{\max(H_f(\sigma_1),\, 0)+2\epsilon}) > \sigma_2. \end{aligned} \tag{125}$$

In other words, if $z \in \Delta$, $s \in C_1^*(\epsilon)$, and if $\delta_2 > 0$ is fixed, then for $|\mathrm{Arg}(e^z - e^s)|$ sufficiently large:

$$|\varphi[\log(e^z - e^s)]| < N(\delta_2)\,|\mathrm{Arg}(e^z - e^s)|^{\mu+\delta_2}, \tag{126}$$

where $\mathrm{Arg}(e^z - e^s)$ is well defined, since by the expansion we gave to $(e^z - e^s)^{-\lambda_n}$ it is seen that $\log(e^z - e^s)$ is to be taken real for z and s real, and it is everywhere (for each couple z and s) defined by analytic continuation. From (125), and from $|\mathcal{R}(s)| < M_2$ and the fact that Δ is bounded, it follows immediately that there exists a constant A such that, if $z \in \Delta$, $s \in C_1^*(\epsilon)$, then $|\mathrm{Arg}(e^z - e^s)| < A\,|\mathrm{Arg}\, e^s| = A|t|$, for $|t|$ sufficiently large:

$$|\varphi[\log(e^z - e^s)]| \leqq N_2(\delta_2)\,|t|^{\mu+\delta_2}.$$

On the other hand, we have on $C_1^*(\epsilon)$ for $|t|$ sufficiently large, and $\delta_1 > 0$ fixed:

$$|f(s)| < N_1(\delta_1)\,|t|^{\nu+\delta_1},$$

and if δ_1 and δ_2 are chosen sufficiently small, in order to have $\mu + \nu + \delta_1 + \delta_2 < k$, the integral in (119) has a meaning

when $z \in L$. It defines there a holomorphic function of z. Since ϵ and ϵ_1 are arbitrarily small this achieves the proof of our theorem.

In another form this theorem was given by the author [8].

A more precise form of this theorem can be obtained in the same way as Theorem XXXIV was obtained from Theorem XXXIII.

THEOREM XLI. *If $s=0$ is a regular point for $f(s)$, then, with the same notations and the same hypotheses as in Theorem XL, the function $F_k^*(s)=\sum d_n e^{-l_n s}$ is uniform in the half-plane $\sigma>\log(e^{Hf(\sigma_1)}+e^{\sigma_2})$, and the only possible singularities of $F_k^*(s)-I_k^*(s)$ where,*

$$I_k^*(s)=f(0)\frac{d^k\varphi[\log(e^s-e^\xi)]}{d\xi^k}_{\xi=0}$$

$$+C_k^1 f'(0)\frac{d^{k-1}\varphi[\log(e^s-e^\xi)]}{d\xi^{k-1}}_{\xi=0}$$

$$+\cdots+f^{(k)}(0)\varphi[\log(e^s-1)],$$

in the half-plane $\sigma>\log(e^{Hf(\sigma_1)}+e^{\sigma_2})$ are the points γ of the form $\gamma=\log(e^\alpha+e^\beta)$, where α is a point of $S_f^{\sigma_1}$, β a point of $S_\varphi^{\sigma_2}$ and the limit-points of this set. The logarithm in $\log(e^\alpha+e^\beta)$ should be taken with all its determinations.

The reader has only to recall how Theorem XXXIV follows from Theorem XXXIII in order to understand how Theorem XLI follows from Theorem XL.

The theorems of this chapter could probably be applied to Dirichlet series with coefficients satisfying some arithmetical properties, in the same way as Hurwitz' theorem was applied in order to furnish such theorems for Dirichlet series (see, for instance [6]).

S. MANDELBROJT.

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